Supplement Material for Manuscript “Ultrafast Dynamics of Strongly Correlated Fermions — Nonequilibrium Green Functions and Selfenergy Approximations”

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1. Derivations in full notation

For the following derivations we repeat several equations from the main text. From Hedin's equations (or the respective scheme for the bare interaction) we rewrite the Dyson equation [main text: Eq. (91)],

$$G_{ij}\Big(z_1, z_2\Big) = G^{(0)}_{ij}\Big(z_1, z_2\Big) + \\ + \int_{\mathcal{C}} \mathrm{d}z_3 \mathrm{d}z_4 \sum_{mn} G^{(0)}_{im}\Big(z_1, z_3\Big)\Sigma_{mn}\Big(z_3, z_4\Big)G_{nj}\Big(z_4, z_2\Big)\,, \tag{1}$$

the selfenergy [main text: Eqs. (95), (97) and (101)],

$$\Sigma_{ij}\Big(z_1, z_2\Big) = \Sigma^{\mathrm{H}}_{ij}\Big(z_1, z_2\Big) + \Sigma^{\mathrm{xc}}_{ij}\Big(z_1, z_2\Big)\,, \tag{2}$$

$$\Sigma^{\mathrm{xc}}_{ij}\Big(z_1, z_2\Big) = \mathrm{i}\hbar \sum_{mpq} w_{ipqm}\Big(z_1\Big) \int_{\mathcal{C}} \mathrm{d}z_3 \sum_{n} G_{mn}\Big(z_1, z_3\Big)\Lambda_{nqpj}\Big(z_3, z_2, z_1\Big)\,, \tag{3}$$

$$\Sigma^{\mathrm{xc}}_{ij}\Big(z_1, z_2\Big) = \mathrm{i}\hbar \int_{\mathcal{C}} \mathrm{d}z_3 \sum_{mpq} W_{ipqm}\Big(z_1, z_3\Big) \times \\ \int_{\mathcal{C}} \mathrm{d}z_4 \sum_{n} G_{mn}\Big(z_1, z_4\Big)\Gamma_{nqpj}\Big(z_4, z_2, z_3\Big)\,, \tag{4}$$

the interaction [main text: Eqs. (102)-(104)],

$$W_{ijkl}\Big(z_1, z_2\Big) = W^{\mathrm{bare}}_{ijkl}\Big(z_1, z_2\Big) + W^{\mathrm{ns}}_{ijkl}\Big(z_1, z_2\Big)\,, \tag{5}$$

$$W^{(1)}_{ijkl}\Big(z_1, z_2\Big) = W^{\mathrm{bare}}_{ijkl}\Big(z_1, z_2\Big) = \delta_{\mathcal{C}}\Big(z_1, z_2\Big)w_{ijkl}\Big(z_1\Big)\,, \tag{6}$$

$$W^{\mathrm{ns}}_{ijkl}\big(z_1,z_2\big)=\sum_{mn}w_{imnl}\big(z_1\big)\int_{\mathcal{C}}\mathrm{d}z_3\sum_{pq}P_{nqpm}\big(z_1,z_3\big)W_{pjkq}\big(z_3,z_2\big)\,, \tag{7}$$

and the polarizability [main text: Eq. (105)],

$$\begin{aligned}P_{ijkl}\big(z_1,z_2\big)=\pm\mathrm{i}\hbar\int_{\mathcal{C}}\mathrm{d}z_3\sum_m G_{im}\big(z_1,z_3\big)\times\\ \int_{\mathcal{C}}\mathrm{d}z_4\sum_n G_{nl}\big(z_4,z_1\big)\Gamma_{mjkn}\big(z_3,z_4,z_2\big)\,.\end{aligned} \tag{8}$$

The first-order selfenergy reads [main text: Eq. (158)]

$$\Sigma^{(1)}_{ij}\big(z_1,z_2\big)=\Sigma^{\mathrm{H}}_{ij}\big(z_1,z_2\big)+\Sigma^{\mathrm{F}}_{ij}\big(z_1,z_2\big)\,, \tag{9}$$

with [main text: Eq. (156)]

$$\begin{aligned}\Sigma^{\mathrm{F}}_{ij}\big(z_1,z_2\big)&=\Sigma^{\mathrm{xc}}_{ij}\Big(W^{(1)}\equiv W^{\mathrm{bare}},\Gamma^{(0)}\Big)=\\ &=\mathrm{i}\hbar\delta_{\mathcal{C}}\big(z_1,z_2\big)\sum_{mn}w_{injm}\big(z_1\big)G_{mn}\big(z_1,z_{1^+}\big)\,.\end{aligned} \tag{10}$$

The zeroth-order screened vertex becomes [main text: Eq. (155)]

$$\Gamma^{(0)}_{ijkl}\big(z_1,z_2,z_3\big)=\delta_{\mathcal{C}}\big(z_1,z_{2^+}\big)\delta_{\mathcal{C}}\big(z_3,z_2\big)\delta_{ik}\delta_{jl}\,. \tag{11}$$

1.1. Second-order selfenergy contributions

1.1.1. Direct second-order selfenergy

The first second-order selfenergy term involves $W^{(2)}$, which structurally is given by, cf. Eq. (5),

$$W^{(2)}=W^{\mathrm{ns}}\Big(P^{(0)},W^{(1)}\Big)\,. \tag{12}$$

The structure of the zeroth-order term of the polarization is

$$P^{(0)}=P\Big(\Gamma^{(0)}\Big)\,. \tag{13}$$

Thus, it is given by

$$\begin{aligned}P^{(0)}_{ijkl}\big(z_1,z_2\big)&=\pm\mathrm{i}\hbar\int_{\mathcal{C}}\mathrm{d}z_3\sum_m G_{im}\big(z_1,z_3\big)\\ &\int_{\mathcal{C}}\mathrm{d}z_4\sum_n G_{nl}\big(z_4,z_1\big)\delta_{\mathcal{C}}\big(z_3,z_{4^+}\big)\delta_{\mathcal{C}}\big(z_2,z_4\big)\delta_{mk}\delta_{jn}\\ &=\pm\mathrm{i}\hbar G_{ik}\big(z_1,z_2\big)G_{jl}\big(z_2,z_1\big)\,.\end{aligned} \tag{14}$$

Inserting this result into Eq. (12), one arrives at

$$W^{(2)}_{ijkl}\Big(z_1, z_2\Big) = \sum_{mn} w_{imnl}\Big(z_1\Big) \int_{\mathcal{C}} \mathrm{d}z_3 \sum_{pq} \Big(\pm \mathrm{i}\hbar G_{np}\Big(z_1, z_3\Big) G_{qm}\Big(z_3, z_1\Big)\Big) \delta_{\mathcal{C}}\Big(z_3, z_2\Big) w_{pjkq}\Big(z_3\Big) \tag{15}$$

and, employing Eq. (7), finally, one has

$$W^{(2)}_{ijkl}\Big(z_1, z_2\Big) = \pm \mathrm{i}\hbar \sum_{mn} w_{imnl}\Big(z_1\Big) \sum_{pq} G_{np}\Big(z_1, z_2\Big) G_{qm}\Big(z_2, z_1\Big) w_{pjkq}\Big(z_2\Big) . \tag{16}$$

With this, $\Sigma^{(2),2,0}$ can be calculated as, cf. Eq. (4),

$$\Sigma^{(2),2,0}_{ij}\Big(z_1, z_2\Big) = \mathrm{i}\hbar \int_{\mathcal{C}} \mathrm{d}z_3 \sum_{mpq} \left(\pm \mathrm{i}\hbar \sum_{rs} w_{irsm}\Big(z_1\Big) \sum_{tu} G_{st}\Big(z_1, z_3\Big) G_{ur}\Big(z_3, z_1\Big) w_{tpqu}\Big(z_3\Big)\right) \int_{\mathcal{C}} \mathrm{d}z_4 \sum_{n} G_{mn}\Big(z_1, z_4\Big) \delta_{\mathcal{C}}\Big(z_4, z_{2^+}\Big) \delta_{\mathcal{C}}\Big(z_3, z_2\Big) \delta_{np} \delta_{qj} . \tag{17}$$

Evaluating the terms, one arrives at

$$\Sigma^{(2),2,0}_{ij}\Big(z_1, z_2\Big) = \pm \Big(\mathrm{i}\hbar\Big)^2 \sum_{mn} G_{mn}\Big(z_1, z_2\Big) \sum_{st} G_{st}\Big(z_1, z_2\Big) \sum_{r} w_{irsm}\Big(z_1\Big) \sum_{u} w_{tnju}\Big(z_2\Big) G_{ur}\Big(z_2, z_1\Big) . \tag{18}$$

1.1.2. Exchange–correlation second-order selfenergy

The other second-order selfenergy term, $\Sigma^{(2),1,1}_{ij}$, requires the first-order term of the vertex Γ, the structure of which is

$$\Gamma^{(1)} = \Gamma\Big(\delta\Sigma^{\mathrm{xc},(1)}/\delta G, \Gamma^{(0)}\Big) . \tag{19}$$

This term involves the functional derivative of $\Sigma^{\mathrm{xc},(1)}$ with respect to G. One has

$$\frac{\delta\Sigma^{\mathrm{xc},(1)}_{ij}\Big(z_1, z_2\Big)}{\delta G_{rs}\Big(z_5, z_6\Big)} = \frac{\delta\Sigma^{\mathrm{xc},(1),\mathrm{F}}_{ij}\Big(z_1, z_2\Big)}{\delta G_{rs}\Big(z_5, z_6\Big)} . \tag{20}$$

Employing Eq. (10), one finds

$$\frac{\delta\Sigma^{\mathrm{xc},(1)}_{ij}\Big(z_1, z_2\Big)}{\delta G_{rs}\Big(z_5, z_6\Big)} = \mathrm{i}\hbar \delta_{\mathcal{C}}\Big(z_1, z_2\Big) \sum_{mn} w_{injm}\Big(z_1\Big) \frac{\delta G_{mn}\Big(z_1, z_{1^+}\Big)}{\delta G_{rs}\Big(z_5, z_6\Big)} = \mathrm{i}\hbar \delta_{\mathcal{C}}\Big(z_1, z_2\Big) \delta_{\mathcal{C}}\Big(z_1, z_5\Big) \delta_{\mathcal{C}}\Big(z_1, z_6\Big) w_{isjr}\Big(z_1\Big) , \tag{21}$$

where

$$\frac{\delta G_{ij}\Big(z_1, z_2\Big)}{\delta G_{mn}\Big(z_5, z_6\Big)} = \delta_{\mathcal{C}}\Big(z_1, z_5\Big)\delta_{\mathcal{C}}\Big(z_2, z_6\Big)\delta_{im}\delta_{jn} \tag{22}$$

has been applied. With this,

$$\begin{aligned} &\Gamma^{(1)}_{ijkl}\Big(z_1, z_2, z_3\Big) \\ &= \int_{\mathcal{C}} \mathrm{d}z_4 \mathrm{d}z_5 \sum_{mn} \frac{\delta \Sigma^{\mathrm{xc},(1)}_{il}\Big(z_1, z_2\Big)}{\delta G_{mn}\Big(z_4, z_5\Big)} \int_{\mathcal{C}} \mathrm{d}z_6 \sum_{p} G_{mp}\Big(z_4, z_6\Big) \\ &\quad \int_{\mathcal{C}} \mathrm{d}z_7 \sum_{q} G_{qn}\Big(z_7, z_5\Big)\Gamma^{(0)}_{pjkq}\Big(z_6, z_7, z_3\Big) . \end{aligned} \tag{23}$$

Using Eq. (21), one has

$$\begin{aligned} &\Gamma^{(1)}_{ijkl}\Big(z_1, z_2, z_3\Big) \\ &= \mathrm{i}\hbar \delta_{\mathcal{C}}\Big(z_1, z_2\Big) \sum_{mn} w_{inlm}\Big(z_1\Big) \int_{\mathcal{C}} \mathrm{d}z_6 \sum_{p} G_{mp}\Big(z_1, z_6\Big) \\ &\quad \int_{\mathcal{C}} \mathrm{d}z_7 \sum_{q} G_{qn}\Big(z_7, z_1\Big)\Gamma^{(0)}_{pjkq}\Big(z_6, z_7, z_3\Big) . \end{aligned} \tag{24}$$

With Eq. (11), finally,

$$\begin{aligned} &\Gamma^{(1)}_{ijkl}\Big(z_1, z_2, z_3\Big) \\ &= \mathrm{i}\hbar \delta_{\mathcal{C}}\Big(z_1, z_2\Big) \sum_{mn} w_{inlm}\Big(z_1\Big) G_{mk}\Big(z_1, z_3\Big) G_{jn}\Big(z_3, z_1\Big) \end{aligned} \tag{25}$$

ensues. Inserting this result yields

$$\begin{aligned} &\Sigma^{(2),1,1}_{ij}\Big(z_1, z_2\Big) \\ &= \mathrm{i}\hbar \int_{\mathcal{C}} \mathrm{d}z_3 \sum_{mpq} W^{(1)}_{ipqm}\Big(z_1, z_3\Big) \int_{\mathcal{C}} \mathrm{d}z_4 \sum_{n} G_{mn}\Big(z_1, z_4\Big)\Gamma^{(1)}_{nqpj}\Big(z_4, z_2, z_3\Big) . \end{aligned} \tag{26}$$

Employing Eqs. (6) and (25), one arrives at

$$\begin{aligned} &\Sigma^{(2),1,1}_{ij}\Big(z_1, z_2\Big) \\ &= \mathrm{i}\hbar \sum_{mpq} w_{ipqm}\Big(z_1\Big) \int_{\mathcal{C}} \mathrm{d}z_4 \sum_{n} G_{mn}\Big(z_1, z_4\Big)\Gamma^{(1)}_{nqpj}\Big(z_4, z_2, z_1\Big) \\ &= \Big(\mathrm{i}\hbar\Big)^2 \sum_{mpq} w_{ipqm}\Big(z_1\Big) \sum_{n} G_{mn}\Big(z_1, z_2\Big) \sum_{rs} w_{nsjr}\Big(z_2\Big) G_{rp}\Big(z_2, z_1\Big) G_{qs}\Big(z_1, z_2\Big) . \end{aligned} \tag{27}$$

1.2. Third-order selfenergy contributions

1.2.1. Third-order term: $\Sigma_{ij}^{(3),\{3;0,2\},0}$

Using Eq. (7), one finds

$$W_{ijkl}^{(3),0,2}\big(z_1,z_2\big) \tag{28}$$
$$=\sum_{mn} w_{imnl}\big(z_1\big)\int_{\mathcal{C}}\mathrm{d}z_3\sum_{pq}\Big(\pm\mathrm{i}\hbar G_{np}\big(z_1,z_3\big)G_{qm}\big(z_3,z_1\big)\Big)W_{pjkq}^{(2)}\big(z_3,z_2\big)\,.$$

Employing Eq. (15) yields

$$W_{ijkl}^{(3),0,2}\big(z_1,z_2\big) \tag{29}$$
$$=\sum_{mn} w_{imnl}\big(z_1\big)\int_{\mathcal{C}}\mathrm{d}z_3\sum_{pq}\Big(\pm\mathrm{i}\hbar G_{np}\big(z_1,z_3\big)G_{qm}\big(z_3,z_1\big)\Big)$$
$$\left(\pm\mathrm{i}\hbar\sum_{rs} w_{prsq}\big(z_3\big)\sum_{tu}G_{st}\big(z_3,z_2\big)G_{ur}\big(z_2,z_3\big)w_{tjku}\big(z_2\big)\right).$$

Evalutating and reordering, one has

$$W_{ijkl}^{(3),0,2}\big(z_1,z_2\big) \tag{30}$$
$$=\big(\mathrm{i}\hbar\big)^2\sum_{mn} w_{imnl}\big(z_1\big)\int_{\mathcal{C}}\mathrm{d}z_3\sum_{pq}G_{np}\big(z_1,z_3\big)G_{qm}\big(z_3,z_1\big)\sum_{rs}w_{prsq}\big(z_3\big)$$
$$\sum_{tu}G_{st}\big(z_3,z_2\big)G_{ur}\big(z_2,z_3\big)w_{tjku}\big(z_2\big)\,.$$

With this, the first term of the first third-order selfenergy class, $\Sigma^{(3),3,0}$, becomes

$$\Sigma_{ij}^{(3),\{3;0,2\},0}\big(z_1,z_2\big) \tag{31}$$
$$=\mathrm{i}\hbar\int_{\mathcal{C}}\mathrm{d}z_3\sum_{mpq}W_{ipqm}^{(3),0,2}\big(z_1,z_3\big)\int_{\mathcal{C}}\mathrm{d}z_4\sum_{n}G_{mn}\big(z_1,z_4\big)\Gamma_{nqpj}^{(0)}\big(z_4,z_2,z_3\big)\,.$$

Using Eq. (11), one has

$$\Sigma_{ij}^{(3),\{3;0,2\},0}\big(z_1,z_2\big) \tag{32}$$
$$=\big(\mathrm{i}\hbar\big)^3\sum_{mn}G_{mn}\big(z_1,z_2\big)\sum_{rs}w_{irsm}\big(z_1\big)\int_{\mathcal{C}}\mathrm{d}z_3\sum_{tu}G_{st}\big(z_1,z_3\big)G_{ur}\big(z_3,z_1\big)$$
$$\sum_{vw}w_{tvwu}\big(z_3\big)\sum_{xy}G_{wx}\big(z_3,z_2\big)G_{yv}\big(z_2,z_3\big)w_{xnjy}\big(z_2\big)\,.$$

1.2.2. Third-order term: $\Sigma_{ij}^{(3),\{3;1,1\},0}$

For the second class of the interaction, $W^{(3),1,1}$, the first-order contribution to the polarizability is needed, which is given by, cf. Eqs. (8) and (25),

$$P_{ijkl}^{(1)}\big(z_1,z_2\big)=P_{ijkl}\big(z_1,z_2\big)\big(\Gamma^{(1)}\big) \tag{33}$$
$$=\pm\mathrm{i}\hbar\int_{\mathcal{C}}\mathrm{d}z_3\sum_{m}G_{im}\big(z_1,z_3\big)\int_{\mathcal{C}}\mathrm{d}z_4\sum_{n}G_{nl}\big(z_4,z_1\big)\Gamma_{mjkn}^{(1)}\big(z_3,z_4,z_2\big)\,.$$

Employing Eq. (25), one arrives at

$$
\begin{aligned}
&P^{(1)}_{ijkl}\big(z_1,z_2\big) \qquad (34)\\
&\quad = \pm\big(\mathrm{i}\hbar\big)^2 \int_{\mathcal{C}} \mathrm{d}z_3 \sum_m G_{im}\big(z_1,z_3\big) \sum_n G_{nl}\big(z_3,z_1\big)\\
&\qquad \sum_{pq} w_{mqnp}\big(z_3\big) G_{pk}\big(z_3,z_2\big) G_{jq}\big(z_2,z_3\big)\,.
\end{aligned}
$$

Inserting this result back, one finds, using Eq. (7),

$$
\begin{aligned}
W^{(3),1,1}_{ijkl}\big(z_1,z_2\big) &= \sum_{mn} w_{imnl}\big(z_1\big) \int_{\mathcal{C}} \mathrm{d}z_3 \sum_{pq} P^{(1)}_{nqpm}\big(z_1,z_3\big) W^{(1)}_{pjkq}\big(z_3,z_2\big)\\
&= \sum_{mn} w_{imnl}\big(z_1\big) \int_{\mathcal{C}} \mathrm{d}z_3 \sum_{pq} \Bigg(\pm\big(\mathrm{i}\hbar\big)^2 \int_{\mathcal{C}} \mathrm{d}z_4 \sum_r G_{nr}\big(z_1,z_4\big) \sum_s G_{sm}\big(z_4,z_1\big)\\
&\quad \sum_{tu} w_{rust}\big(z_4\big) G_{tp}\big(z_4,z_3\big) G_{qu}\big(z_3,z_4\big)\Bigg) \delta_{\mathcal{C}}\big(z_3,z_2\big) w_{pjkq}\big(z_2\big)\,. \qquad (35)
\end{aligned}
$$

After reordering, one has

$$
\begin{aligned}
&W^{(3),1,1}_{ijkl}\big(z_1,z_2\big) \qquad (36)\\
&\quad = \pm\big(\mathrm{i}\hbar\big)^2 \sum_{mn} w_{imnl}\big(z_1\big) \int_{\mathcal{C}} \mathrm{d}z_3 \sum_r G_{nr}\big(z_1,z_3\big) \sum_s G_{sm}\big(z_3,z_1\big)\\
&\qquad \sum_{tu} w_{rust}\big(z_3\big) \sum_{pq} G_{tp}\big(z_3,z_2\big) G_{qu}\big(z_2,z_3\big) w_{pjkq}\big(z_2\big)\,.
\end{aligned}
$$

With these results, the second term of the class $\Sigma^{(3),3,0}$ is found, using Eq. (4),

$$
\begin{aligned}
&\Sigma^{(3),\{3;1,1\},0}_{ij}\big(z_1,z_2\big) \qquad (37)\\
&\quad = \mathrm{i}\hbar \int_{\mathcal{C}} \mathrm{d}z_3 \sum_{mpq} W^{(3),1,1}_{ipqm}\big(z_1,z_3\big) \int_{\mathcal{C}} \mathrm{d}z_4 \sum_n G_{mn}\big(z_1,z_4\big) \Gamma^{(0)}_{nqpj}\big(z_4,z_2,z_3\big)\,.
\end{aligned}
$$

Using Eq. (11), one finds

$$
\begin{aligned}
&\Sigma^{(3),\{3;1,1\},0}_{ij}\big(z_1,z_2\big) \qquad (38)\\
&\quad = \pm\big(\mathrm{i}\hbar\big)^3 \sum_{mn} G_{mn}\big(z_1,z_2\big) \sum_{rs} w_{irsm}\big(z_1\big) \int_{\mathcal{C}} \mathrm{d}z_3 \sum_t G_{st}\big(z_1,z_3\big) \sum_u G_{ur}\big(z_3,z_1\big)\\
&\qquad \sum_{vw} w_{twuv}\big(z_3\big) \sum_{xy} G_{vx}\big(z_3,z_2\big) G_{yw}\big(z_2,z_3\big) w_{xnjy}\big(z_2\big)\,.
\end{aligned}
$$

1.2.3. Third-order term: $\Sigma^{(3),2,1}_{ij}$

Continuing with the second class $\Sigma^{(3),2,1}$, it is directly worked out by combining Eqs. (15)

and (25),

$$\Sigma^{(3),2,1}_{ij}\left(z_1,z_2\right) = \mathrm{i}\hbar \int_{\mathcal{C}} \mathrm{d}z_3 \sum_{mpq} W^{(2)}_{ipqm}\left(z_1,z_3\right) \int_{\mathcal{C}} \mathrm{d}z_4 \sum_{n} G_{mn}\left(z_1,z_4\right)\Gamma^{(1)}_{nqpj}\left(z_4,z_2,z_3\right) . \tag{39}$$

Inserting Eq. (25) yields

$$\begin{aligned}\Sigma^{(3),2,1}_{ij}\left(z_1,z_2\right) &= \pm\left(\mathrm{i}\hbar\right)^3 \int_{\mathcal{C}} \mathrm{d}z_3 \sum_{mrs} w_{irsm}\left(z_1\right) \sum_{tu} G_{st}\left(z_1,z_3\right)G_{ur}\left(z_3,z_1\right) \sum_{pq} w_{tpqu}\left(z_3\right) \\ &\quad \sum_{n} G_{mn}\left(z_1,z_2\right) \sum_{vw} w_{nwjv}\left(z_2\right)G_{vp}\left(z_2,z_3\right)G_{qw}\left(z_3,z_2\right) .\end{aligned} \tag{40}$$

1.2.4. Third-order term: $\Sigma^{(3),1,\{2;1,1\}}_{ij}$

For the single contribution to the class $\Gamma^{(2),1,1}$, one finds, employing Eqs. (21) and (25),

$$\begin{aligned}\Gamma^{(2),1,1}_{ijkl}\left(z_1,z_2,z_3\right) &= \int_{\mathcal{C}} \mathrm{d}z_4 \mathrm{d}z_5 \sum_{mn} \frac{\delta\Sigma^{\mathrm{xc},(1)}_{il}\left(z_1,z_2\right)}{\delta G_{mn}\left(z_4,z_5\right)} \int_{\mathcal{C}} \mathrm{d}z_6 \sum_{p} G_{mp}\left(z_4,z_6\right) \\ &\quad \int_{\mathcal{C}} \mathrm{d}z_7 \sum_{q} G_{qn}\left(z_7,z_5\right)\Gamma^{(1)}_{pjkq}\left(z_6,z_7,z_3\right) .\end{aligned} \tag{41}$$

Evaluating the derivative yields

$$\begin{aligned}\Gamma^{(2),1,1}_{ijkl}\left(z_1,z_2,z_3\right) &= \mathrm{i}\hbar\delta_{\mathcal{C}}\left(z_1,z_2\right) \sum_{mn} w_{inlm}\left(z_1\right) \int_{\mathcal{C}} \mathrm{d}z_6 \sum_{p} G_{mp}\left(z_1,z_6\right) \\ &\quad \int_{\mathcal{C}} \mathrm{d}z_7 \sum_{q} G_{qn}\left(z_7,z_1\right)\Gamma^{(1)}_{pjkq}\left(z_6,z_7,z_3\right) .\end{aligned} \tag{42}$$

Employing Eq. (25), one finds

$$\begin{aligned}\Gamma^{(2),1,1}_{ijkl}\left(z_1,z_2,z_3\right) &= \left(\mathrm{i}\hbar\right)^2\delta_{\mathcal{C}}\left(z_1,z_2\right) \sum_{mn} w_{inlm}\left(z_1\right) \int_{\mathcal{C}} \mathrm{d}z_6 \sum_{p} G_{mp}\left(z_1,z_6\right) \\ &\quad \sum_{q} G_{qn}\left(z_6,z_1\right) \sum_{rs} w_{psqr}\left(z_6\right)G_{rk}\left(z_6,z_3\right)G_{js}\left(z_3,z_6\right) .\end{aligned} \tag{43}$$

This enables the computation of $\Sigma^{(3),1,\{2;1,1\}}$ with Eqs. (4) and (6),

$$\begin{aligned}\Sigma^{(3),1,\{2;1,1\}}_{ij}\Big(z_1, z_2\Big) \\ &= \mathrm{i}\hbar \int_{\mathcal{C}} \mathrm{d}z_3 \sum_{mpq} W^{(1)}_{ipqm}\Big(z_1, z_3\Big) \int_{\mathcal{C}} \mathrm{d}z_4 \sum_{n} G_{mn}\Big(z_1, z_4\Big) \Gamma^{(2),1,1}_{nqpj}\Big(z_4, z_2, z_3\Big) \\ &= \mathrm{i}\hbar \sum_{mpq} w_{ipqm}\Big(z_1\Big) \int_{\mathcal{C}} \mathrm{d}z_4 \sum_{n} G_{mn}\Big(z_1, z_4\Big) \Gamma^{(2),1,1}_{nqpj}\Big(z_4, z_2, z_1\Big) .\end{aligned} \tag{44}$$

Using Eq. (43), one arrives at

$$\begin{aligned}\Sigma^{(3),1,\{2;1,1\}}_{ij}\Big(z_1, z_2\Big) \\ &= \Big(\mathrm{i}\hbar\Big)^3 \sum_{mpq} w_{ipqm}\Big(z_1\Big) \sum_{n} G_{mn}\Big(z_1, z_2\Big) \sum_{rs} w_{nsjr}\Big(z_2\Big) \int_{\mathcal{C}} \mathrm{d}z_3 \sum_{t} G_{rt}\Big(z_2, z_3\Big) \\ &\quad \sum_{u} G_{us}\Big(z_3, z_2\Big) \sum_{vw} w_{twuv}\Big(z_3\Big) G_{vp}\Big(z_3, z_1\Big) G_{qw}\Big(z_1, z_3\Big) .\end{aligned} \tag{45}$$

1.2.5. Second-order vertex terms

For the first terms, one finds

$$\begin{aligned}\Gamma^{(2),\{2;2,0\},0}_{ijkl}\Big(z_1, z_2, z_3\Big) &= \int_{\mathcal{C}} \mathrm{d}z_4 \mathrm{d}z_5 \sum_{mn} \frac{\delta \Sigma^{(2),2,0}_{il}\Big(z_1, z_2\Big)}{\delta G_{mn}\Big(z_4, z_5\Big)} \int_{\mathcal{C}} \mathrm{d}z_6 \sum_{p} G_{mp}\Big(z_4, z_6\Big) \\ &\quad \int_{\mathcal{C}} \mathrm{d}z_7 \sum_{q} G_{qn}\Big(z_7, z_5\Big) \Gamma^{(0)}_{pjkq}\Big(z_6, z_7, z_3\Big) \\ &= \int_{\mathcal{C}} \mathrm{d}z_4 \mathrm{d}z_5 \sum_{mn} \frac{\delta \Sigma^{(2),2,0}_{il}\Big(z_1, z_2\Big)}{\delta G_{mn}\Big(z_4, z_5\Big)} G_{mk}\Big(z_4, z_3\Big) G_{jn}\Big(z_3, z_5\Big) .\end{aligned} \tag{46}$$

Inserting Eq. (18), the term attains the form

$$\begin{aligned}\Gamma^{(2),\{2;2,0\},0}_{ijkl}\Big(z_1, z_2, z_3\Big) &= \pm\Big(\mathrm{i}\hbar\Big)^2 \int_{\mathcal{C}} \mathrm{d}z_4 \mathrm{d}z_5 \\ &\quad \sum_{mn} \frac{\delta\Big(\sum_{pq} G_{pq}\Big(z_1, z_2\Big) \sum_{st} G_{st}\Big(z_1, z_2\Big) \sum_{r} w_{irsp}\Big(z_1\Big) \sum_{u} w_{tqlu}\Big(z_2\Big) G_{ur}\Big(z_2, z_1\Big)\Big)}{\delta G_{mn}\Big(z_4, z_5\Big)} \\ &\quad G_{mk}\Big(z_4, z_3\Big) G_{jn}\Big(z_3, z_5\Big) .\end{aligned} \tag{47}$$

Evaluating the derivative, one has

$$\begin{aligned}\Gamma^{(2),\{2;2,0\},0}_{ijkl}\Big(z_1, z_2, z_3\Big) &= \Gamma^{(2),\{2;2,0\},0,\mathrm{A}}_{ijkl}\Big(z_1, z_2, z_3\Big) \\ &\quad + \Gamma^{(2),\{2;2,0\},0,\mathrm{B}}_{ijkl}\Big(z_1, z_2, z_3\Big) + \Gamma^{(2),\{2;2,0\},0,\mathrm{C}}_{ijkl}\Big(z_1, z_2, z_3\Big) ,\end{aligned} \tag{48}$$

with

$$\Gamma^{(2),\{2;2,0\},0,\mathrm{A}}_{ijkl}\left(z_1,z_2,z_3\right) \tag{49}$$
$$= \pm\left(\mathrm{i}\hbar\right)^2 \sum_{mn}\sum_{st} G_{st}\left(z_1,z_2\right)\sum_r w_{irsm}\left(z_1\right)$$
$$\sum_u w_{tnlu}\left(z_2\right)G_{ur}\left(z_2,z_1\right)G_{mk}\left(z_1,z_3\right)G_{jn}\left(z_3,z_2\right)$$

and

$$\Gamma^{(2),\{2;2,0\},0,\mathrm{B}}_{ijkl}\left(z_1,z_2,z_3\right) \tag{50}$$
$$\pm\left(\mathrm{i}\hbar\right)^2 \sum_{mn}\sum_{pq} G_{pq}\left(z_1,z_2\right)\sum_r w_{irmp}\left(z_1\right)$$
$$\sum_u w_{nqlu}\left(z_2\right)G_{ur}\left(z_2,z_1\right)G_{mk}\left(z_1,z_3\right)G_{jn}\left(z_3,z_2\right)$$

as well as

$$\Gamma^{(2),\{2;2,0\},0,\mathrm{C}}_{ijkl}\left(z_1,z_2,z_3\right) \tag{51}$$
$$\pm\left(\mathrm{i}\hbar\right)^2 \sum_{mn}\sum_{pq} G_{pq}\left(z_1,z_2\right)\sum_{st} G_{st}\left(z_1,z_2\right)$$
$$w_{insp}\left(z_1\right)w_{tqlm}\left(z_2\right)G_{mk}\left(z_2,z_3\right)G_{jn}\left(z_3,z_1\right).$$

Similarly, one finds

$$\Gamma^{(2),\{2;1,1\},0}_{ijkl}\left(z_1,z_2,z_3\right) = \int_{\mathcal{C}}\mathrm{d}z_4\mathrm{d}z_5\,\sum_{mn}\frac{\delta\Sigma^{(2),1,1}_{il}\left(z_1,z_2\right)}{\delta G_{mn}\left(z_4,z_5\right)}\int_{\mathcal{C}}\mathrm{d}z_6\,\sum_p G_{mp}\left(z_4,z_6\right)$$
$$\int_{\mathcal{C}}\mathrm{d}z_7\,\sum_q G_{qn}\left(z_7,z_5\right)\Gamma^{(0)}_{pjkq}\left(z_6,z_7,z_3\right). \tag{52}$$

Inserting Eq. (27) yields

$$\Gamma^{(2),\{2;1,1\},0}_{ijkl}\left(z_1,z_2,z_3\right) = \left(\mathrm{i}\hbar\right)^2\int_{\mathcal{C}}\mathrm{d}z_4\mathrm{d}z_5 \tag{53}$$
$$\sum_{mn}\frac{\delta\left(\sum_{prs} w_{irsp}\left(z_1\right)\sum_q G_{pq}\left(z_1,z_2\right)\sum_{tu} w_{qult}\left(z_2\right)G_{tr}\left(z_2,z_1\right)G_{su}\left(z_1,z_2\right)\right)}{\delta G_{mn}\left(z_4,z_5\right)}$$
$$G_{mk}\left(z_4,z_3\right)G_{jn}\left(z_3,z_5\right),$$

which, after evaluation of the derivative, yields

$$\Gamma^{(2),\{2;1,1\},0}_{ijkl}\left(z_1,z_2,z_3\right) = \Gamma^{(2),\{2;1,1\},0,\mathrm{A}}_{ijkl}\left(z_1,z_2,z_3\right) \tag{54}$$
$$+\,\Gamma^{(2),\{2;1,1\},0,\mathrm{B}}_{ijkl}\left(z_1,z_2,z_3\right)+\Gamma^{(2),\{2;1,1\},0,\mathrm{C}}_{ijkl}\left(z_1,z_2,z_3\right),$$

with

$$\Gamma^{(2),\{2;1,1\},0,\mathrm{A}}_{ijkl}\Big(z_1,z_2,z_3\Big) \tag{55}$$
$$= \Big(\mathrm{i}\hbar\Big)^2 \sum_{mn}\sum_{rs} w_{irsm}\Big(z_1\Big) \sum_{tu} w_{nult}\Big(z_2\Big) G_{tr}\Big(z_2,z_1\Big)$$
$$G_{su}\Big(z_1,z_2\Big) G_{mk}\Big(z_1,z_3\Big) G_{jn}\Big(z_3,z_2\Big)$$

and

$$\Gamma^{(2),\{2;1,1\},0,\mathrm{B}}_{ijkl}\Big(z_1,z_2,z_3\Big) \tag{56}$$
$$+ \Big(\mathrm{i}\hbar\Big)^2 \sum_{mn}\sum_{ps} w_{insp}\Big(z_1\Big) \sum_{q} G_{pq}\Big(z_1,z_2\Big) \sum_{u} w_{qulm}\Big(z_2\Big)$$
$$G_{su}\Big(z_1,z_2\Big) G_{mk}\Big(z_2,z_3\Big) G_{jn}\Big(z_3,z_1\Big)$$

as well as

$$\Gamma^{(2),\{2;1,1\},0,\mathrm{C}}_{ijkl}\Big(z_1,z_2,z_3\Big) \tag{57}$$
$$+ \Big(\mathrm{i}\hbar\Big)^2 \sum_{mn}\sum_{pr} w_{irmp}\Big(z_1\Big) \sum_{q} G_{pq}\Big(z_1,z_2\Big) \sum_{t} w_{qnlt}\Big(z_2\Big)$$
$$G_{tr}\Big(z_2,z_1\Big) G_{mk}\Big(z_1,z_3\Big) G_{jn}\Big(z_3,z_2\Big)\,.$$

1.2.6. Third-order terms: $\Sigma^{(3),1,2}_{ij}$

With this result, the corresponding selfenergy terms can be computed,

$$\Sigma^{(3),1,\{2;\{2;2,0\},0,\mathrm{A}\}}_{ij}\Big(z_1,z_2\Big) \tag{58}$$
$$= \mathrm{i}\hbar \int_{\mathcal{C}} \mathrm{d}z_3 \sum_{mpq} W^{(1)}_{ipqm}\Big(z_1,z_3\Big) \int_{\mathcal{C}} \mathrm{d}z_4 \sum_{n} G_{mn}\Big(z_1,z_4\Big) \Gamma^{(2),\{2;2,0\},0,\mathrm{A}}_{nqpj}\Big(z_4,z_2,z_3\Big)$$
$$= \mathrm{i}\hbar \sum_{mpq} w_{ipqm}\Big(z_1\Big) \int_{\mathcal{C}} \mathrm{d}z_4 \sum_{n} G_{mn}\Big(z_1,z_4\Big) \Gamma^{(2),\{2;2,0\},0,\mathrm{A}}_{nqpj}\Big(z_4,z_2,z_1\Big)\,.$$

Inserting Eq. (49), one arrives at

$$\Sigma^{(3),1,\{2;\{2;2,0\},0,\mathrm{A}\}}_{ij}\Big(z_1,z_2\Big) = \pm\Big(\mathrm{i}\hbar\Big)^3 \sum_{mpq} w_{ipqm}\Big(z_1\Big) \tag{59}$$
$$\int_{\mathcal{C}} \mathrm{d}z_4 \sum_{n} G_{mn}\Big(z_1,z_4\Big) \sum_{rsuv} G_{uv}\Big(z_4,z_2\Big) \sum_{t} w_{ntur}\Big(z_4\Big)$$
$$\sum_{w} w_{vsjw}\Big(z_2\Big) G_{wt}\Big(z_2,z_4\Big) G_{rp}\Big(z_4,z_1\Big) G_{qs}\Big(z_1,z_2\Big)\,.$$

For the second term, one has

$$\Sigma^{(3),1,\{2;\{2;2,0\},0,\mathrm{B}\}}_{ij}\Big(z_1,z_2\Big) \tag{60}$$
$$= \mathrm{i}\hbar \sum_{mpq} w_{ipqm}\Big(z_1\Big) \int_{\mathcal{C}} \mathrm{d}z_4 \sum_{n} G_{mn}\Big(z_1,z_4\Big) \Gamma^{(2),\{2;2,0\},0,\mathrm{B}}_{nqpj}\Big(z_4,z_2,z_1\Big)\,.$$

Using Eq. (50) yields

$$\begin{aligned}
&\Sigma^{(3),1,\{2;\{2;2,0\},0,\mathrm{B}\}}_{ij}\Big(z_1,z_2\Big) \\
&\quad= \pm\Big(\mathrm{i}\hbar\Big)^3 \sum_{mpq} w_{ipqm}\Big(z_1\Big)\int_{\mathcal{C}} \mathrm{d}z_4 \sum_n G_{mn}\Big(z_1,z_4\Big)\sum_{rs}\sum_{tu} G_{tu}\Big(z_4,z_2\Big) \\
&\qquad \sum_v w_{nvrt}\Big(z_4\Big)\sum_w w_{sujw}\Big(z_2\Big)G_{wv}\Big(z_2,z_4\Big)G_{rp}\Big(z_4,z_1\Big)G_{qs}\Big(z_1,z_2\Big)\,.
\end{aligned} \tag{61}$$

The third term is given by

$$\begin{aligned}
&\Sigma^{(3),1,\{2;\{2;2,0\},0,\mathrm{C}\}}_{ij}\Big(z_1,z_2\Big) \\
&\quad= \mathrm{i}\hbar \sum_{mpq} w_{ipqm}\Big(z_1\Big)\int_{\mathcal{C}} \mathrm{d}z_4 \sum_n G_{mn}\Big(z_1,z_4\Big)\Gamma^{(2),\{2;2,0\},0,\mathrm{C}}_{nqpj}\Big(z_4,z_2,z_1\Big)\,.
\end{aligned} \tag{62}$$

With Eq. (51), one finds

$$\begin{aligned}
&\Sigma^{(3),1,\{2;\{2;2,0\},0,\mathrm{C}\}}_{ij}\Big(z_1,z_2\Big) \\
&\quad= \pm\Big(\mathrm{i}\hbar\Big)^3 \sum_{mpq} w_{ipqm}\Big(z_1\Big)\int_{\mathcal{C}} \mathrm{d}z_4 \sum_n G_{mn}\Big(z_1,z_4\Big)\sum_{rs}\sum_{tu} G_{tu}\Big(z_4,z_2\Big) \\
&\qquad \sum_{vw} G_{vw}\Big(z_4,z_2\Big)w_{nsvt}\Big(z_4\Big)w_{wujr}\Big(z_2\Big)G_{rp}\Big(z_2,z_1\Big)G_{qs}\Big(z_1,z_4\Big)\,.
\end{aligned} \tag{63}$$

For the other class, one has

$$\begin{aligned}
&\Sigma^{(3),1,\{2;\{2;1,1\},0,\mathrm{A}\}}_{ij}\Big(z_1,z_2\Big) \\
&\quad= \mathrm{i}\hbar \int_{\mathcal{C}} \mathrm{d}z_3 \sum_{mpq} W^{(1)}_{ipqm}\Big(z_1,z_3\Big)\int_{\mathcal{C}} \mathrm{d}z_4 \sum_n G_{mn}\Big(z_1,z_4\Big)\Gamma^{(2),\{2;1,1\},0,\mathrm{A}}_{nqpj}\Big(z_4,z_2,z_3\Big) \\
&\quad= \mathrm{i}\hbar \sum_{mpq} w_{ipqm}\Big(z_1\Big)\int_{\mathcal{C}} \mathrm{d}z_4 \sum_n G_{mn}\Big(z_1,z_4\Big)\Gamma^{(2),1,1,\mathrm{A}}_{nqpj}\Big(z_4,z_2,z_1\Big)\,.
\end{aligned} \tag{64}$$

Inserting Eq. (55) yields

$$\begin{aligned}
&\Sigma^{(3),1,\{2;\{2;1,1\},0,\mathrm{A}\}}_{ij}\Big(z_1,z_2\Big) \\
&\quad= \Big(\mathrm{i}\hbar\Big)^3 \sum_{mpq} w_{ipqm}\Big(z_1\Big)\int_{\mathcal{C}} \mathrm{d}z_4 \sum_n G_{mn}\Big(z_1,z_4\Big)\sum_{rs}\sum_{tu} w_{ntur}\Big(z_4\Big) \\
&\qquad \sum_{vw} w_{swjv}\Big(z_2\Big)G_{vt}\Big(z_2,z_4\Big)G_{uw}\Big(z_4,z_2\Big)G_{rp}\Big(z_4,z_1\Big)G_{qs}\Big(z_1,z_2\Big)\,.
\end{aligned} \tag{65}$$

Similarly, the second term reads

$$\begin{aligned}
&\Sigma^{(3),1,\{2;\{2;1,1\},0,\mathrm{B}\}}_{ij}\Big(z_1,z_2\Big) \\
&\quad= \mathrm{i}\hbar \sum_{mpq} w_{ipqm}\Big(z_1\Big)\int_{\mathcal{C}} \mathrm{d}z_4 \sum_n G_{mn}\Big(z_1,z_4\Big)\Gamma^{(2),\{2;1,1\},0,\mathrm{B}}_{nqpj}\Big(z_4,z_2,z_1\Big)\,.
\end{aligned} \tag{66}$$

With Eq. (56), one has

$$\begin{aligned}
&\Sigma_{ij}^{(3),1,\{2;\{2;1,1\},0,\mathrm{B}\}}\left(z_1,z_2\right) \qquad (67)\\
&\quad=\left(\mathrm{i}\hbar\right)^3\sum_{mpq}w_{ipqm}\left(z_1\right)\int_{\mathcal{C}}\mathrm{d}z_4\,\sum_n G_{mn}\left(z_1,z_4\right)\sum_{rs}\sum_{tv}w_{nsvt}\left(z_4\right)\\
&\qquad\sum_u G_{tu}\left(z_4,z_2\right)\sum_w w_{uwjr}\left(z_2\right)G_{vw}\left(z_4,z_2\right)G_{rp}\left(z_2,z_1\right)G_{qs}\left(z_1,z_4\right).
\end{aligned}$$

For the third term, one finds

$$\begin{aligned}
&\Sigma_{ij}^{(3),1,\{2;\{2;1,1\},0,\mathrm{C}\}}\left(z_1,z_2\right) \qquad (68)\\
&\quad=\mathrm{i}\hbar\sum_{mpq}w_{ipqm}\left(z_1\right)\int_{\mathcal{C}}\mathrm{d}z_4\,\sum_n G_{mn}\left(z_1,z_4\right)\Gamma_{nqpj}^{(2),\{2;1,1\},0,\mathrm{C}}\left(z_4,z_2,z_1\right).
\end{aligned}$$

Employing Eq. (57), one arrives at

$$\begin{aligned}
&\Sigma_{ij}^{(3),1,\{2;\{2;1,1\},0,\mathrm{C}\}}\left(z_1,z_2\right) \qquad (69)\\
&\quad=\left(\mathrm{i}\hbar\right)^3\sum_{mpq}w_{ipqm}\left(z_1\right)\int_{\mathcal{C}}\mathrm{d}z_4\,\sum_n G_{mn}\left(z_1,z_4\right)\sum_{rs}\sum_{tv}w_{nvrt}\left(z_4\right)\\
&\qquad\sum_u G_{tu}\left(z_4,z_2\right)\sum_w w_{usjw}\left(z_2\right)G_{wv}\left(z_2,z_4\right)G_{rp}\left(z_4,z_1\right)G_{qs}\left(z_1,z_2\right).
\end{aligned}$$

1.3. Resummation approaches: GW approximation

The GW approximation solves Hedin's equation for the screened interaction W according to Eq. (7) with the zeroth-order vertex $\Gamma^{(0)}$. The set of equations is given by the Dyson equation, cf. Eq. (1),

$$\begin{aligned}
G_{ij}\left(z_1,z_2\right)&=G_{ij}^{(0)}\left(z_1,z_2\right) \qquad (70)\\
&\quad+\int_{\mathcal{C}}\mathrm{d}z_3\mathrm{d}z_4\,\sum_{mn}G_{im}^{(0)}\left(z_1,z_3\right)\Sigma_{mn}\left(z_3,z_4\right)G_{nj}\left(z_4,z_2\right),
\end{aligned}$$

the equation for the selfenergy [cf. Eq. (2)]

$$\Sigma_{ij}\left(z_1,z_2\right)=\Sigma_{ij}^{\mathrm{H}}\left(z_1,z_2\right)+\Sigma_{ij}^{\mathrm{xc}}\left(z_1,z_2\right), \qquad (71)$$

with

$$\begin{aligned}
\Sigma_{ij}^{\mathrm{xc}}\left(z_1,z_2\right)&=\mathrm{i}\hbar\int_{\mathcal{C}}\mathrm{d}z_3\,\sum_{mpq}W_{ipqm}\left(z_1,z_3\right) \qquad (72)\\
&\quad\int_{\mathcal{C}}\mathrm{d}z_4\,\sum_n G_{mn}\left(z_1,z_4\right)\Gamma_{nqpj}^{(0)}\left(z_4,z_2,z_3\right)\\
&\quad=\mathrm{i}\hbar\sum_{mp}W_{ipjm}\left(z_1,z_2\right)G_{mp}\left(z_1,z_2\right),
\end{aligned}$$

the zeroth-order polarizability, cf. Eq. (14),

$$P_{ijkl}\big(z_1, z_2\big) = P^{(0)}_{ijkl}\big(z_1, z_2\big) = \pm\mathrm{i}\hbar G_{ik}\big(z_1, z_2\big) G_{jl}\big(z_2, z_1\big)\,, \tag{73}$$

the zeroth-order vertex, cf. Eq. (11),

$$\Gamma_{ijkl}\big(z_1, z_2, z_3\big) = \Gamma^{(0)}_{ijkl}\big(z_1, z_2, z_3\big) = \delta_{\mathcal{C}}\big(z_1, z_{2^+}\big)\delta_{\mathcal{C}}\big(z_3, z_2\big)\delta_{ik}\delta_{jl} \tag{74}$$

and the screened interaction [cf. Eqs. (5) and (7)]

$$W_{ijkl}\big(z_1, z_2\big) = \delta_{\mathcal{C}}\big(z_1, z_2\big) w_{ijkl}\big(z_1\big) + W^{\mathrm{ns}}_{ijkl}\big(z_1, z_2\big)\,, \tag{75}$$

with

$$\begin{aligned} &W^{\mathrm{ns}}_{ijkl}\big(z_1, z_2\big) \\ &\quad= \sum_{mn} w_{imnl}\big(z_1\big) \int_{\mathcal{C}} \mathrm{d}z_3 \sum_{pq} P^{(0)}_{nqpm}\big(z_1, z_3\big) W_{pjkq}\big(z_3, z_2\big) \\ &\quad= \pm\mathrm{i}\hbar \sum_{mn} w_{imnl}\big(z_1\big) \int_{\mathcal{C}} \mathrm{d}z_3 \sum_{pq} G_{np}\big(z_1, z_3\big) G_{qm}\big(z_3, z_1\big) W_{pjkq}\big(z_3, z_2\big)\,. \end{aligned} \tag{76}$$

To solve this set of equations, one has to determine the selfconsistent solution of Eq. (75). Thereto, it is more suitable to eliminate the singular bare interaction by using

$$\begin{aligned} &W^{\mathrm{ns}}_{ijkl}\big(z_1, z_2\big) = W_{ijkl}\big(z_1, z_2\big) - W^{\mathrm{bare}}_{ijkl}\big(z_1, z_2\big) \\ &= W_{ijkl}\big(z_1, z_2\big) - \delta_{\mathcal{C}}\big(z_1, z_2\big) w_{ijkl}\big(z_1\big)\,. \end{aligned} \tag{77}$$

The selfenergy [cf. Eqs. (71) and (72)] in terms of W^{ns} is then given by

$$\begin{aligned} \Sigma^{\mathrm{GW}}_{ij}\big(z_1, z_2\big) &= \Sigma^{\mathrm{H}}_{ij}\big(z_1, z_2\big) + \mathrm{i}\hbar \sum_{mp} W_{ipjm}\big(z_1, z_2\big) G_{mp}\big(z_1, z_2\big) \\ &= \Sigma^{\mathrm{H}}_{ij}\big(z_1, z_2\big) + \mathrm{i}\hbar \sum_{mp} w_{ipjm}\big(z_1\big) G_{mp}\big(z_1, z_{1^+}\big)\delta_{\mathcal{C}}\big(z_1, z_2\big) \\ &\quad + \mathrm{i}\hbar \sum_{mp} W^{\mathrm{ns}}_{ipjm}\big(z_1, z_2\big) G_{mp}\big(z_1, z_2\big)\,. \end{aligned} \tag{78}$$

Using Eq. (10), the expression simplifies to

$$\begin{aligned} &\Sigma^{\mathrm{GW}}_{ij}\big(z_1, z_2\big) \\ &\quad= \Sigma^{\mathrm{H}}_{ij}\big(z_1, z_2\big) + \Sigma^{\mathrm{F}}_{ij}\big(z_1, z_2\big) + \mathrm{i}\hbar \sum_{mp} W^{\mathrm{ns}}_{ipjm}\big(z_1, z_2\big) G_{mp}\big(z_1, z_2\big) \\ &\quad=: \Sigma^{\mathrm{H}}_{ij}\big(z_1, z_2\big) + \Sigma^{\mathrm{F}}_{ij}\big(z_1, z_2\big) + \Sigma^{\mathrm{GW,corr}}_{ij}\big(z_1, z_2\big)\,. \end{aligned} \tag{79}$$

For the screened interaction, one has

$$
\begin{aligned}
&W^{\mathrm{ns}}_{ijkl}\big(z_1,z_2\big) \\
&\quad= \pm\mathrm{i}\hbar\sum_{mn} w_{imnl}\big(z_1\big)\sum_{pq} G_{np}\big(z_1,z_2\big)G_{qm}\big(z_2,z_1\big)w_{pjkq}\big(z_2\big) \\
&\qquad\pm\mathrm{i}\hbar\sum_{mn} w_{imnl}\big(z_1\big)\int_{\mathcal{C}}\mathrm{d}z_3\sum_{pq} G_{np}\big(z_1,z_3\big)G_{qm}\big(z_3,z_1\big)W^{\mathrm{ns}}_{pjkq}\big(z_3,z_2\big)\,.
\end{aligned}
\tag{80}
$$

1.4. Resummation approaches: T matrix

In contrast to the GW approximation, the T matrix is an approximation, which takes only the bare interaction w into account and aims instead at a good approximation of the bare vertex function Λ. Thus, its constitutive equations are the Dyson equation,

$$
\begin{aligned}
&G_{ij}\big(z_1,z_2\big) = \\
&\quad G^{(0)}_{ij}\big(z_1,z_2\big) + \int_{\mathcal{C}}\mathrm{d}z_3\mathrm{d}z_4\sum_{mn} G^{(0)}_{im}\big(z_1,z_3\big)\Sigma_{mn}\big(z_3,z_4\big)G_{nj}\big(z_4,z_2\big)\,,
\end{aligned}
\tag{81}
$$

the equation for the selfenergy, cf. Eqs. (2) and (3),

$$
\Sigma_{ij}\big(z_1,z_2\big) = \Sigma^{\mathrm{H}}_{ij}\big(z_1,z_2\big) + \Sigma^{\mathrm{xc}}_{ij}\big(z_1,z_2\big)\,,
\tag{82}
$$

with

$$
\begin{aligned}
&\Sigma^{\mathrm{xc}}_{ij}\big(z_1,z_2\big) \\
&\quad= \mathrm{i}\hbar\sum_{mpq} w_{ipqm}\big(z_1\big)\int_{\mathcal{C}}\mathrm{d}z_3\sum_{n} G_{mn}\big(z_1,z_3\big)\Lambda_{nqpj}\big(z_3,z_2,z_1\big)\,.
\end{aligned}
\tag{83}
$$

The bare vertex Λ is self-consistently given as the solution of

$$
\begin{aligned}
&\Lambda_{ijkl}\big(z_1,z_2,z_3\big) = \delta_{\mathcal{C}}\big(z_1,z_{2+}\big)\delta_{\mathcal{C}}\big(z_3,z_2\big)\delta_{ik}\delta_{jl} \\
&\quad+\int_{\mathcal{C}}\mathrm{d}z_4\mathrm{d}z_5\sum_{mn}\frac{\delta\Sigma_{il}\big(z_1,z_2\big)}{\delta G_{mn}\big(z_4,z_5\big)}\int_{\mathcal{C}}\mathrm{d}z_6\sum_{p} G_{mp}\big(z_4,z_6\big) \\
&\quad\int_{\mathcal{C}}\mathrm{d}z_7\sum_{q} G_{qn}\big(z_7,z_5\big)\Lambda_{pjkq}\big(z_6,z_7,z_3\big)\,.
\end{aligned}
\tag{84}
$$

If these equations are iterated ad infinitum, all selfenergy terms will be generated. To break the circular dependence between Eqs. (83) and (84), the T-matrix approximation starts by taking the bare vertex on the right-hand side of Eq. (84) only in zeroth order,

$$
\Lambda^{(0)}_{ijkl}\big(z_1,z_2,z_3\big) = \delta_{\mathcal{C}}\big(z_1,z_{2+}\big)\delta_{\mathcal{C}}\big(z_3,z_2\big)\delta_{ik}\delta_{jl}\,,
\tag{85}
$$

transforming it into

$$\Lambda^{\mathrm{cl}}_{ijkl}\big(z_1,z_2,z_3\big) = \delta_{\mathcal{C}}\big(z_1,z_{2^+}\big)\delta_{\mathcal{C}}\big(z_3,z_2\big)\delta_{ik}\delta_{jl} + \int_{\mathcal{C}} \mathrm{d}z_4\mathrm{d}z_5 \sum_{mn} \frac{\delta\Sigma^{\mathrm{cl}}_{il}\big(z_1,z_2\big)}{\delta G_{mn}\big(z_4,z_5\big)} G_{mk}\big(z_4,z_3\big)G_{jn}\big(z_3,z_5\big)\,. \tag{86}$$

To arrive at a closed equation, this result is used in Eq. (82), yielding

$$\begin{aligned}\Sigma^{\mathrm{cl}}_{ij}\big(z_1,z_2\big) &= \Sigma^{\mathrm{H}}_{ij}\big(z_1,z_2\big) + \Sigma^{\mathrm{xc}}_{ij}\big(z_1,z_2\big) \\ &= \Sigma^{\mathrm{H}}_{ij}\big(z_1,z_2\big) + \mathrm{i}\hbar \sum_{mpq} w_{ipqm}\big(z_1\big) \int_{\mathcal{C}} \mathrm{d}z_3 \sum_{n} G_{mn}\big(z_1,z_3\big)\Lambda^{\mathrm{cl}}_{nqpj}\big(z_3,z_2,z_1\big)\,.\end{aligned} \tag{87}$$

Inserting Eq. (86), one has

$$\begin{aligned}\Sigma^{\mathrm{cl}}_{ij}\big(z_1,z_2\big) &= \pm\mathrm{i}\hbar\delta_{\mathcal{C}}\big(z_1,z_2\big)\sum_{mn} w_{mijn}\big(z_1\big)G_{nm}\big(z_1,z_{1^+}\big) \\ &\quad + \mathrm{i}\hbar\sum_{mn} w_{injm}\big(z_1\big)G_{mn}\big(z_1,z_{1^+}\big)\delta_{\mathcal{C}}\big(z_1,z_2\big) \\ &\quad + \mathrm{i}\hbar\sum_{mpq} w_{ipqm}\big(z_1\big)\int_{\mathcal{C}}\mathrm{d}z_3 \sum_n G_{mn}\big(z_1,z_3\big) \\ &\quad \int_{\mathcal{C}}\mathrm{d}z_4\mathrm{d}z_5 \sum_{rs} \frac{\delta\Sigma^{\mathrm{cl}}_{nj}\big(z_3,z_2\big)}{\delta G_{rs}\big(z_4,z_5\big)} G_{rp}\big(z_4,z_1\big)G_{qs}\big(z_1,z_5\big)\,.\end{aligned} \tag{88}$$

This term can be restructured to yield

$$\begin{aligned}&\Sigma^{\mathrm{cl}}_{ij}\big(z_1,z_2\big) \\ &= \mathrm{i}\hbar\delta_{\mathcal{C}}\big(z_1,z_2\big)\sum_{mn} G_{mn}\big(z_1,z_{1^+}\big)w^{\pm}_{injm}\big(z_1\big) \\ &\quad + \mathrm{i}\hbar\sum_{mpq} w_{ipqm}\big(z_1\big)\int_{\mathcal{C}}\mathrm{d}z_3 \sum_n G_{mn}\big(z_1,z_3\big) \\ &\quad \int_{\mathcal{C}}\mathrm{d}z_4\mathrm{d}z_5 \sum_{rs} \frac{\delta\Sigma^{\mathrm{cl}}_{nj}\big(z_3,z_2\big)}{\delta G_{rs}\big(z_4,z_5\big)} G_{rp}\big(z_4,z_1\big)G_{qs}\big(z_1,z_5\big)\,,\end{aligned} \tag{89}$$

with the (anti-)symmetrized potential $w^{\pm}_{ijkl}\big(z_1\big) := w_{ijkl}\big(z_1\big) \pm w_{jikl}\big(z_1\big)$. Using Eq. (10) again, one finds

$$\begin{aligned}&\Sigma^{\mathrm{cl}}_{ij}\big(z_1,z_2\big) \\ &= \Sigma^{\mathrm{H}}_{ij}\big(z_1,z_2\big) + \Sigma^{\mathrm{F}}_{ij}\big(z_1,z_2\big) + \mathrm{i}\hbar\sum_{mpq} w_{ipqm}\big(z_1\big)\int_{\mathcal{C}}\mathrm{d}z_3 \sum_n G_{mn}\big(z_1,z_3\big) \\ &\quad \int_{\mathcal{C}}\mathrm{d}z_4\mathrm{d}z_5 \sum_{rs} \frac{\delta\Sigma^{\mathrm{cl}}_{nj}\big(z_3,z_2\big)}{\delta G_{rs}\big(z_4,z_5\big)} G_{rp}\big(z_4,z_1\big)G_{qs}\big(z_1,z_5\big)\,.\end{aligned} \tag{90}$$

Taking the derivative with respect to G, one arrives at

$$\frac{\delta\Sigma^{\mathrm{cl}}_{ij}\left(z_1,z_2\right)}{\delta G_{tu}\left(z_6,z_7\right)} = \mathrm{i}\hbar\delta_{\mathcal{C}}\left(z_1,z_2\right)\delta_{\mathcal{C}}\left(z_1,z_6\right)\delta_{\mathcal{C}}\left(z_{1^+},z_7\right)w^{\pm}_{iujt}\left(z_1\right) \left(\frac{\delta\Sigma^{\mathrm{cl,A}}_{ij}\left(z_1,z_2\right)}{\delta G_{tu}\left(z_6,z_7\right)} + \frac{\delta\Sigma^{\mathrm{cl,B}}_{ij}\left(z_1,z_2\right)}{\delta G_{tu}\left(z_6,z_7\right)} + \frac{\delta\Sigma^{\mathrm{cl,C}}_{ij}\left(z_1,z_2\right)}{\delta G_{tu}\left(z_6,z_7\right)} + \frac{\delta\Sigma^{\mathrm{cl,D}}_{ij}\left(z_1,z_2\right)}{\delta G_{tu}\left(z_6,z_7\right)}\right), \tag{91}$$

with

$$\frac{\delta\Sigma^{\mathrm{cl,A}}_{ij}\left(z_1,z_2\right)}{\delta G_{tu}\left(z_6,z_7\right)} = \mathrm{i}\hbar\delta_{\mathcal{C}}\left(z_1,z_6\right)\sum_{pq} w_{ipqt}\left(z_1\right) \int_{\mathcal{C}}\mathrm{d}z_4\mathrm{d}z_5 \sum_{rs}\frac{\delta\Sigma^{\mathrm{cl}}_{uj}\left(z_7,z_2\right)}{\delta G_{rs}\left(z_4,z_5\right)}G_{rp}\left(z_4,z_1\right)G_{qs}\left(z_1,z_5\right), \tag{92}$$

$$\frac{\delta\Sigma^{\mathrm{cl,B}}_{ij}\left(z_1,z_2\right)}{\delta G_{tu}\left(z_6,z_7\right)} = \mathrm{i}\hbar\delta_{\mathcal{C}}\left(z_1,z_7\right)\sum_{mq} w_{iuqm}\left(z_1\right)\int_{\mathcal{C}}\mathrm{d}z_3\sum_{n}G_{mn}\left(z_1,z_3\right) \int_{\mathcal{C}}\mathrm{d}z_5\sum_{s}\frac{\delta\Sigma^{\mathrm{cl}}_{nj}\left(z_3,z_2\right)}{\delta G_{ts}\left(z_6,z_5\right)}G_{qs}\left(z_1,z_5\right) \tag{93}$$

and

$$\frac{\delta\Sigma^{\mathrm{cl,C}}_{ij}\left(z_1,z_2\right)}{\delta G_{tu}\left(z_6,z_7\right)} = \mathrm{i}\hbar\delta_{\mathcal{C}}\left(z_1,z_6\right)\sum_{mp} w_{iptm}\left(z_1\right)\int_{\mathcal{C}}\mathrm{d}z_3\sum_{n}G_{mn}\left(z_1,z_3\right) \int_{\mathcal{C}}\mathrm{d}z_4\sum_{r}\frac{\delta\Sigma^{\mathrm{cl}}_{nj}\left(z_3,z_2\right)}{\delta G_{ru}\left(z_4,z_7\right)}G_{rp}\left(z_4,z_1\right), \tag{94}$$

as well as

$$\frac{\delta\Sigma^{\mathrm{cl,D}}_{ij}\left(z_1,z_2\right)}{\delta G_{tu}\left(z_6,z_7\right)} \tag{95}$$
$$= \mathrm{i}\hbar\sum_{mpq} w_{ipqm}\left(z_1\right)\int_{\mathcal{C}}\mathrm{d}z_3\,\sum_n G_{mn}\left(z_1,z_3\right)$$
$$\int_{\mathcal{C}}\mathrm{d}z_4\mathrm{d}z_5\,\sum_{rs}\frac{\delta\Sigma^{\mathrm{cl}}_{nj}\left(z_3,z_2\right)}{\delta G_{rs}\left(z_4,z_5\right)\delta G_{tu}\left(z_6,z_7\right)}G_{rp}\left(z_4,z_1\right)G_{qs}\left(z_1,z_5\right).$$

Neglecting $\frac{\delta\Sigma^{\mathrm{cl}}_{nj}\left(z_3,z_2\right)}{\delta G_{rs}\left(z_4,z_5\right)\delta G_{tu}\left(z_6,z_7\right)}$ as an approximation, Eq. (91) becomes a closed equation for $\frac{\delta\Sigma^{\mathrm{cl}}}{\delta G}$. The first iteration yields the second-order terms

$$\frac{\delta\Sigma^{\mathrm{cl},(2)}_{ij}\left(z_1,z_2\right)}{\delta G_{tu}\left(z_6,z_7\right)} \approx \frac{\delta\Sigma^{\mathrm{cl},(2),\mathrm{A}}_{ij}\left(z_1,z_2\right)}{\delta G_{tu}\left(z_6,z_7\right)} \tag{96}$$
$$+\frac{\delta\Sigma^{\mathrm{cl},(2),\mathrm{B}}_{ij}\left(z_1,z_2\right)}{\delta G_{tu}\left(z_6,z_7\right)}+\frac{\delta\Sigma^{\mathrm{cl},(2),\mathrm{C}}_{ij}\left(z_1,z_2\right)}{\delta G_{tu}\left(z_6,z_7\right)},$$

with

$$\frac{\delta\Sigma^{\mathrm{cl},(2),\mathrm{A}}_{ij}\left(z_1,z_2\right)}{\delta G_{tu}\left(z_6,z_7\right)} \tag{97}$$
$$= \left(\mathrm{i}\hbar\right)^2\delta_{\mathcal{C}}\left(z_1,z_6\right)\delta_{\mathcal{C}}\left(z_2,z_7\right)\sum_{pq} w_{ipqt}\left(z_1\right)$$
$$\sum_{rs} w^{\pm}_{usjr}\left(z_2\right)G_{rp}\left(z_2,z_1\right)G_{qs}\left(z_1,z_2\right)$$

and

$$\frac{\delta\Sigma^{\mathrm{cl},(2),\mathrm{B}}_{ij}\left(z_1,z_2\right)}{\delta G_{tu}\left(z_6,z_7\right)} \tag{98}$$
$$= \left(\mathrm{i}\hbar\right)^2\delta_{\mathcal{C}}\left(z_1,z_7\right)\delta_{\mathcal{C}}\left(z_2,z_6\right)\sum_{mq} w_{iuqm}\left(z_1\right)$$
$$\sum_n G_{mn}\left(z_1,z_2\right)\sum_s w^{\pm}_{nsjt}\left(z_2\right)G_{qs}\left(z_1,z_2\right)$$

as well as

$$\frac{\delta\Sigma^{\mathrm{cl},(2),\mathrm{C}}_{ij}\left(z_1,z_2\right)}{\delta G_{tu}\left(z_6,z_7\right)} \tag{99}$$
$$= \left(\mathrm{i}\hbar\right)^2\delta_{\mathcal{C}}\left(z_1,z_6\right)\delta_{\mathcal{C}}\left(z_2,z_7\right)\sum_{mp} w_{iptm}\left(z_1\right)$$
$$\sum_n G_{mn}\left(z_1,z_2\right)\sum_r w^{\pm}_{nujr}\left(z_2\right)G_{rp}\left(z_2,z_1\right).$$

Note that, for the first iteration, $\frac{\delta\Sigma^{\mathrm{cl,D}}_{ij}(z_1,z_2)}{\delta G_{tu}(z_6,z_7)}$ is exactly equal to zero, thus Eq. (96) is also exact up to second order in w. In the following, each of the three terms will be considered separately. To start with, one recognizes that all three terms yield the same first and second-order contributions to the selfenergy, which read

$$\Sigma^{\mathrm{cl},(1)}_{ij}\left(z_1,z_2\right) = \Sigma^{\mathrm{H}}_{ij}\left(z_1,z_2\right) + \Sigma^{\mathrm{F}}_{ij}\left(z_1,z_2\right) \tag{100}$$

$$\Sigma^{\mathrm{cl},(2)}_{ij}\left(z_1,z_2\right) = \left(\mathrm{i}\hbar\right)^2 \sum_{mpq} w_{ipqm}\left(z_1\right) \sum_n G_{mn}\left(z_1,z_2\right) \sum_{rs} w^{\pm}_{nsjr}\left(z_2\right) G_{rp}\left(z_2,z_1\right) G_{qs}\left(z_1,z_2\right) \tag{101}$$

and agree with the exact first and second-order terms, already encountered in Eqs. (9), (18) and (27). The third-order contributions to Σ^{cl} from the second-order terms in Eq. (96) are given by

$$\Sigma^{\mathrm{cl},(3),\mathrm{A}}_{ij}\left(z_1,z_2\right) = \mathrm{i}\hbar \sum_{mpq} w_{ipqm}\left(z_1\right) \int_{\mathcal{C}} \mathrm{d}z_3 \sum_n G_{mn}\left(z_1,z_3\right) \int_{\mathcal{C}} \mathrm{d}z_4 \mathrm{d}z_5 \sum_{rs} \frac{\delta\Sigma^{\mathrm{cl},(2),\mathrm{A}}_{nj}\left(z_3,z_2\right)}{\delta G_{rs}\left(z_4,z_5\right)} G_{rp}\left(z_4,z_1\right) G_{qs}\left(z_1,z_5\right) . \tag{102}$$

Inserting the second-order term yields

$$\Sigma^{\mathrm{cl},(3),\mathrm{A}}_{ij}\left(z_1,z_2\right) = \left(\mathrm{i}\hbar\right)^3 \sum_{mpq} w_{ipqm}\left(z_1\right) \int_{\mathcal{C}} \mathrm{d}z_3 \sum_n G_{mn}\left(z_1,z_3\right) \sum_{rs}\sum_{tu} w_{ntur}\left(z_3\right) \sum_{vw} w^{\pm}_{swjv}\left(z_2\right) G_{vt}\left(z_2,z_3\right) G_{uw}\left(z_3,z_2\right) G_{rp}\left(z_3,z_1\right) G_{qs}\left(z_1,z_2\right) . \tag{103}$$

For the second third-order selfenergy term, one has

$$\Sigma^{\mathrm{cl},(3),\mathrm{B}}_{ij}\left(z_1,z_2\right) = \mathrm{i}\hbar \sum_{mpq} w_{ipqm}\left(z_1\right) \int_{\mathcal{C}} \mathrm{d}z_3 \sum_n G_{mn}\left(z_1,z_3\right) \int_{\mathcal{C}} \mathrm{d}z_4 \mathrm{d}z_5 \sum_{rs} \frac{\delta\Sigma^{\mathrm{cl},(2),\mathrm{B}}_{nj}\left(z_3,z_2\right)}{\delta G_{rs}\left(z_4,z_5\right)} G_{rp}\left(z_4,z_1\right) G_{qs}\left(z_1,z_5\right) . \tag{104}$$

which evaluates to

$$\Sigma^{\mathrm{cl},(3),\mathrm{B}}_{ij}\left(z_1,z_2\right) = \left(\mathrm{i}\hbar\right)^3 \sum_{mpq} w_{ipqm}\left(z_1\right) \int_{\mathcal{C}} \mathrm{d}z_3 \sum_n G_{mn}\left(z_1,z_3\right) \sum_{rs}\sum_{tu} w_{nsut}\left(z_3\right) \sum_v G_{tv}\left(z_3,z_2\right) \sum_w w^{\pm}_{vwjr}\left(z_2\right) G_{uw}\left(z_3,z_2\right) G_{rp}\left(z_2,z_1\right) G_{qs}\left(z_1,z_3\right) . \tag{105}$$

The third term reads

$$\Sigma^{\mathrm{cl},(3),\mathrm{C}}_{ij}\Big(z_1, z_2\Big) = \mathrm{i}\hbar \sum_{mpq} w_{ipqm}\Big(z_1\Big) \int_{\mathcal{C}} \mathrm{d}z_3 \sum_{n} G_{mn}\Big(z_1, z_3\Big) \tag{106}$$

$$\int_{\mathcal{C}} \mathrm{d}z_4 \mathrm{d}z_5 \sum_{rs} \frac{\delta\Sigma^{\mathrm{cl},(2),\mathrm{C}}_{nj}\Big(z_3, z_2\Big)}{\delta G_{rs}\Big(z_4, z_5\Big)} G_{rp}\Big(z_4, z_1\Big) G_{qs}\Big(z_1, z_5\Big) \,. \tag{107}$$

Using the second-order result, one arrives at

$$\begin{aligned}\Sigma^{\mathrm{cl},(3),\mathrm{C}}_{ij}\Big(z_1, z_2\Big) = \mathrm{i}\hbar \sum_{mpq} w_{ipqm}\Big(z_1\Big) \int_{\mathcal{C}} \mathrm{d}z_3 \sum_{n} G_{mn}\Big(z_1, z_3\Big) \\ \sum_{rs}\sum_{tu} w_{nurt}\Big(z_3\Big) \sum_{v} G_{tv}\Big(z_3, z_2\Big) \sum_{w} w^{\pm}_{vsjw}\Big(z_2\Big) \\ G_{wu}\Big(z_2, z_3\Big) G_{rp}\Big(z_3, z_1\Big) G_{qs}\Big(z_1, z_2\Big) \,.\end{aligned} \tag{108}$$

2. Non-selfconsistent second-order selfenergy contributions

2.1. General basis

The first two classes are just the same as in the selfconsistent approximation, cf. Eqs. (176) and (183) of the main text, with the replacement $G \to G^{(0)}$. Likewise, their components follow directly from Eqs. (184) and (185) of the main text. For the third and fourth class, one needs the contributions to $G^{(1)}$, which are

$$\begin{aligned}&G^{(1),\{\mathrm{H},0\},0}_{ij}\Big(z_1, z_2\Big) \\ &= \int_{\mathcal{C}} \mathrm{d}z_3 \mathrm{d}z_4 \sum_{mn} G^{(0)}_{im}\Big(z_1, z_3\Big) \Sigma^{\mathrm{H},0}_{mn}\Big(z_3, z_4\Big) G^{(0)}_{nj}\Big(z_4, z_2\Big) \\ &= \pm\mathrm{i}\hbar \int_{\mathcal{C}} \mathrm{d}z_3 \sum_{mn} G^{(0)}_{im}\Big(z_1, z_3\Big) \sum_{pq} w_{pmnq}\Big(z_3\Big) G^{(0)}_{qp}\Big(z_3, z_{3^+}\Big) G^{(0)}_{nj}\Big(z_3, z_2\Big)\end{aligned} \tag{109}$$

and

$$\begin{aligned}&G^{(1),\{\mathrm{F},0\},0}_{ij}\Big(z_1, z_2\Big) \\ &= \int_{\mathcal{C}} \mathrm{d}z_3 \mathrm{d}z_4 \sum_{mn} G^{(0)}_{im}\Big(z_1, z_3\Big) \Sigma^{\mathrm{F},0}_{mn}\Big(z_3, z_4\Big) G^{(0)}_{nj}\Big(z_4, z_2\Big) \\ &= \mathrm{i}\hbar \int_{\mathcal{C}} \mathrm{d}z_3 \sum_{mn} G^{(0)}_{im}\Big(z_1, z_3\Big) \sum_{pq} w_{mqnp}\Big(z_3\Big) G^{(0)}_{pq}\Big(z_3, z_{3^+}\Big) G^{(0)}_{nj}\Big(z_3, z_2\Big) \,.\end{aligned} \tag{110}$$

With these results, the additional non-selfconsistent contributions are

$$\Sigma_{ij}^{(2),\{\mathrm{H},0\},\{1,\{\mathrm{H},0\},0\}}\Big(z_1,z_2\Big)=\pm\mathrm{i}\hbar\delta_{\mathcal{C}}\Big(z_1,z_2\Big)\sum_{mn}w_{mijn}\Big(z_1\Big)G_{nm}^{(1),\{\mathrm{H},0\},0}\Big(z_1,z_{1^+}\Big)$$
$$=\Big(\mathrm{i}\hbar\Big)^2\delta_{\mathcal{C}}\Big(z_1,z_2\Big)\sum_{mn}w_{mijn}\Big(z_1\Big) \tag{111}$$
$$\int_{\mathcal{C}}\mathrm{d}z_3\sum_{pq}G_{np}^{(0)}\Big(z_1,z_3\Big)\sum_{rs}w_{rpqs}\Big(z_3\Big)G_{sr}^{(0)}\Big(z_3,z_{3^+}\Big)G_{qm}^{(0)}\Big(z_3,z_{1^+}\Big)\,,$$

as well as

$$\Sigma_{ij}^{(2),\{\mathrm{H},0\},\{1,\{\mathrm{F},0\},0\}}\Big(z_1,z_2\Big)=\pm\mathrm{i}\hbar\delta_{\mathcal{C}}\Big(z_1,z_2\Big)\sum_{mn}w_{mijn}\Big(z_1\Big)G_{nm}^{(1),\{\mathrm{F},0\},0}\Big(z_1,z_{1^+}\Big)$$
$$=\pm\Big(\mathrm{i}\hbar\Big)^2\delta_{\mathcal{C}}\Big(z_1,z_2\Big)\sum_{mn}w_{mijn}\Big(z_1\Big) \tag{112}$$
$$\int_{\mathcal{C}}\mathrm{d}z_3\sum_{pq}G_{np}^{(0)}\Big(z_1,z_3\Big)\sum_{rs}w_{psqr}\Big(z_3\Big)G_{rs}^{(0)}\Big(z_3,z_{3^+}\Big)G_{qm}^{(0)}\Big(z_3,z_{1^+}\Big)$$

and

$$\Sigma_{ij}^{(2),\{\mathrm{F},0\},\{1,\{\mathrm{H},0\},0\}}\Big(z_1,z_2\Big)=\mathrm{i}\hbar\delta_{\mathcal{C}}\Big(z_1,z_2\Big)\sum_{mn}w_{injm}\Big(z_1\Big)G_{mn}^{(1),\{\mathrm{H},0\},0}\Big(z_1,z_{1^+}\Big)$$
$$=\pm\Big(\mathrm{i}\hbar\Big)^2\delta_{\mathcal{C}}\Big(z_1,z_2\Big)\sum_{mn}w_{nimj}\Big(z_1\Big) \tag{113}$$
$$\int_{\mathcal{C}}\mathrm{d}z_3\sum_{pq}G_{mp}^{(0)}\Big(z_1,z_3\Big)\sum_{rs}w_{rpqs}\Big(z_3\Big)G_{sr}^{(0)}\Big(z_3,z_{3^+}\Big)G_{qn}^{(0)}\Big(z_3,z_{1^+}\Big)\,,$$

as well as

$$\Sigma_{ij}^{(2),\{\mathrm{F},0\},\{1,\{\mathrm{F},0\},0\}}\Big(z_1,z_2\Big)=\mathrm{i}\hbar\delta_{\mathcal{C}}\Big(z_1,z_2\Big)\sum_{mn}w_{injm}\Big(z_1\Big)G_{mn}^{(1),\{\mathrm{F},0\},0}\Big(z_1,z_{1^+}\Big)$$
$$=\Big(\mathrm{i}\hbar\Big)^2\delta_{\mathcal{C}}\Big(z_1,z_2\Big)\sum_{mn}w_{nimj}\Big(z_1\Big) \tag{114}$$
$$\int_{\mathcal{C}}\mathrm{d}z_3\sum_{pq}G_{mp}^{(0)}\Big(z_1,z_3\Big)\sum_{rs}w_{psqr}\Big(z_3\Big)G_{rs}^{(0)}\Big(z_3,z_{3^+}\Big)G_{qn}^{(0)}\Big(z_3,z_{1^+}\Big)\,.$$

The corresponding components are all time-diagonal and read

$$\Sigma_{ij}^{(2),\{\mathrm{H},0\},\{1,\{\mathrm{H},0\},0\},\delta}\Big(t_1\Big) \tag{115}$$
$$=\Big(\mathrm{i}\hbar\Big)^2\sum_{mn}w_{mijn}\Big(t_1\Big)\Bigg($$
$$\int_{t_0}^{t_1}\mathrm{d}t_3\sum_{pq}G_{np}^{(0),>}\Big(t_1,t_3\Big)\sum_{rs}w_{rpqs}\Big(t_3\Big)G_{sr}^{(0),<}\Big(t_3,t_3\Big)G_{qm}^{(0),<}\Big(t_3,t_1\Big)$$
$$+\int_{t_1}^{t_0}\mathrm{d}t_3\sum_{pq}G_{np}^{(0),<}\Big(t_1,t_3\Big)\sum_{rs}w_{rpqs}\Big(t_3\Big)G_{sr}^{(0),<}\Big(t_3,t_3\Big)G_{qm}^{(0),>}\Big(t_3,t_1\Big)\Bigg)$$

and

$$\begin{aligned}
&\Sigma_{ij}^{(2),\{\mathrm{H},0\},\{1,\{\mathrm{F},0\},0\},\delta}\Big(t_1\Big) \qquad (116)\\
&\quad = \pm\Big(\mathrm{i}\hbar\Big)^2 \sum_{mn} w_{mijn}\Big(t_1\Big)\Bigg(\\
&\quad\quad \int_{t_0}^{t_1} \mathrm{d}t_3 \sum_{pq} G_{np}^{(0),>}\Big(t_1,t_3\Big) \sum_{rs} w_{psqr}\Big(t_3\Big) G_{rs}^{(0),<}\Big(t_3,t_3\Big) G_{qm}^{(0),<}\Big(t_3,t_1\Big)\\
&\quad\quad + \int_{t_1}^{t_0} \mathrm{d}t_3 \sum_{pq} G_{np}^{(0),<}\Big(t_1,t_3\Big) \sum_{rs} w_{psqr}\Big(t_3\Big) G_{rs}^{(0),<}\Big(t_3,t_3\Big) G_{qm}^{(0),>}\Big(t_3,t_1\Big)\Bigg)\,,
\end{aligned}$$

as well as

$$\begin{aligned}
&\Sigma_{ij}^{(2),\{\mathrm{F},0\},\{1,\{\mathrm{H},0\},0\},\delta}\Big(t_1\Big) \qquad (117)\\
&\quad = \pm\Big(\mathrm{i}\hbar\Big)^2 \sum_{mn} w_{nimj}\Big(t_1\Big)\Bigg(\\
&\quad\quad \int_{t_0}^{t_1} \mathrm{d}t_3 \sum_{pq} G_{mp}^{(0),>}\Big(t_1,t_3\Big) \sum_{rs} w_{rpqs}\Big(t_3\Big) G_{sr}^{(0),<}\Big(t_3,t_3\Big) G_{qn}^{(0),<}\Big(t_3,t_1\Big)\\
&\quad\quad + \int_{t_1}^{t_0} \mathrm{d}t_3 \sum_{pq} G_{mp}^{(0),<}\Big(t_1,t_3\Big) \sum_{rs} w_{rpqs}\Big(t_3\Big) G_{sr}^{(0),<}\Big(t_3,t_3\Big) G_{qn}^{(0),>}\Big(t_3,t_1\Big)\Bigg)
\end{aligned}$$

and

$$\begin{aligned}
&\Sigma_{ij}^{(2),\{\mathrm{F},0\},\{1,\{\mathrm{F},0\},0\},\delta}\Big(t_1\Big) \qquad (118)\\
&\quad = \Big(\mathrm{i}\hbar\Big)^2 \sum_{mn} w_{nimj}\Big(t_1\Big)\Bigg(\\
&\quad\quad \int_{t_0}^{t_1} \mathrm{d}t_3 \sum_{pq} G_{mp}^{(0),>}\Big(t_1,t_3\Big) \sum_{rs} w_{psqr}\Big(t_3\Big) G_{rs}^{(0),<}\Big(t_3,t_3\Big) G_{qn}^{(0),<}\Big(t_3,t_1\Big)\\
&\quad\quad + \int_{t_1}^{t_0} \mathrm{d}t_3 \sum_{pq} G_{mp}^{(0),<}\Big(t_1,t_3\Big) \sum_{rs} w_{psqr}\Big(t_3\Big) G_{rs}^{(0),<}\Big(t_3,t_3\Big) G_{qn}^{(0),>}\Big(t_3,t_1\Big)\Bigg)\,.
\end{aligned}$$

2.2. Diagonal basis

For $w_{ijkl} = \delta_{il}\delta_{jk}w_{ij}$, the non-selfconsistent selfenergy terms attain the form, cf. Eqs. (189) and (191) of the main text,

$$\begin{aligned}
&\Sigma_{ij}^{(2),(2),2,0,0,\mathrm{diagonal}}\Big(z_1,z_2\Big) \qquad (119)\\
&\quad \equiv \Sigma_{ij}^{(2),2,0,\mathrm{diagonal}}\Big(z_1,z_2\Big)\Big(G \to G^{(0)}\Big)
\end{aligned}$$

and

$$\begin{aligned}
&\Sigma_{ij}^{(2),(2),1,0,1,\mathrm{diagonal}}\Big(z_1,z_2\Big) \qquad (120)\\
&\quad \equiv \Sigma_{ij}^{(2),1,1,\mathrm{diagonal}}\Big(z_1,z_2\Big)\Big(G \to G^{(0)}\Big)\,,
\end{aligned}$$

as well as

$$\Sigma_{ij}^{(2),\{\mathrm{H},0\},\{1,\{\mathrm{H},0\},0\},\mathrm{diagonal}}\left(z_1,z_2\right)
= \left(\mathrm{i}\hbar\right)^2\delta_{\mathcal{C}}\left(z_1,z_2\right)\delta_{ij}\sum_m w_{mi}\left(z_1\right)
\int_{\mathcal{C}}\mathrm{d}z_3\,\sum_p G_{mp}^{(0)}\left(z_1,z_3\right)G_{pm}^{(0)}\left(z_3,z_{1^+}\right)\sum_r w_{rp}\left(z_3\right)G_{rr}^{(0)}\left(z_3,z_{3^+}\right) \tag{121}$$

and

$$\Sigma_{ij}^{(2),\{\mathrm{H},0\},\{1,\{\mathrm{F},0\},0\},\mathrm{diagonal}}\left(z_1,z_2\right)
= \pm\left(\mathrm{i}\hbar\right)^2\delta_{\mathcal{C}}\left(z_1,z_2\right)\delta_{ij}\sum_m w_{mi}\left(z_1\right)
\int_{\mathcal{C}}\mathrm{d}z_3\,\sum_p G_{mp}^{(0)}\left(z_1,z_3\right)\sum_q w_{pq}\left(z_3\right)G_{pq}^{(0)}\left(z_3,z_{3^+}\right)G_{qm}^{(0)}\left(z_3,z_{1^+}\right). \tag{122}$$

Further,

$$\Sigma_{ij}^{(2),\{\mathrm{F},0\},\{1,\{\mathrm{H},0\},0\},\mathrm{diagonal}}\left(z_1,z_2\right)
= \pm\left(\mathrm{i}\hbar\right)^2\delta_{\mathcal{C}}\left(z_1,z_2\right)w_{ij}\left(z_1\right)
\int_{\mathcal{C}}\mathrm{d}z_3\,\sum_p G_{ip}^{(0)}\left(z_1,z_3\right)\sum_r w_{rp}\left(z_3\right)G_{rr}^{(0)}\left(z_3,z_{3^+}\right)G_{pj}^{(0)}\left(z_3,z_{1^+}\right) \tag{123}$$

and

$$\Sigma_{ij}^{(2),\{\mathrm{F},0\},\{1,\{\mathrm{F},0\},0\},\mathrm{diagonal}}\left(z_1,z_2\right)
= \left(\mathrm{i}\hbar\right)^2\delta_{\mathcal{C}}\left(z_1,z_2\right)w_{ij}\left(z_1\right)
\int_{\mathcal{C}}\mathrm{d}z_3\,\sum_p G_{ip}^{(0)}\left(z_1,z_3\right)\sum_q w_{pq}\left(z_3\right)G_{pq}^{(0)}\left(z_3,z_{3^+}\right)G_{qj}^{(0)}\left(z_3,z_{1^+}\right). \tag{124}$$

The components read [cf. Eqs. (191) and (192) of the main text]

$$\Sigma_{ij}^{(2),(2),2,0,0,\mathrm{diagonal},\gtrless}\left(t_1,t_2\right) \equiv \Sigma_{ij}^{(2),2,0,\mathrm{diagonal},\gtrless}\left(t_1,t_2\right)\left(G\to G^{(0)}\right), \tag{125}$$

$$\Sigma_{ij}^{(2),(2),1,0,1,\mathrm{diagonal},\gtrless}\left(t_1,t_2\right) \equiv \Sigma_{ij}^{(2),1,1,\mathrm{diagonal},\gtrless}\left(t_1,t_2\right)\left(G\to G^{(0)}\right), \tag{126}$$

as well as [cf. Eq. (115) to Eq. (119)]

$$\Sigma_{ij}^{(2),\{\mathrm{H},0\},\{1,\{\mathrm{H},0\},0\},\mathrm{diagonal},\delta}\left(t_1\right)
= \left(\mathrm{i}\hbar\right)^2\delta_{ij}\sum_m w_{mi}\left(t_1\right)\Bigg(
\int_{t_0}^{t_1}\mathrm{d}t_3\,\sum_p G_{mp}^{(0),>}\left(t_1,t_3\right)\sum_r w_{rp}\left(t_3\right)G_{rr}^{(0),<}\left(t_3,t_3\right)G_{pm}^{(0),<}\left(t_3,t_1\right)
+\int_{t_1}^{t_0}\mathrm{d}t_3\,\sum_p G_{mp}^{(0),<}\left(t_1,t_3\right)\sum_r w_{rp}\left(t_3\right)G_{rr}^{(0),<}\left(t_3,t_3\right)G_{pm}^{(0),>}\left(t_3,t_1\right)\Bigg) \tag{127}$$

and

$$\Sigma_{ij}^{(2),\{\mathrm{H},0\},\{1,\{\mathrm{F},0\},0\},\delta}\left(t_1\right) \tag{128}$$
$$= \pm\left(\mathrm{i}\hbar\right)^2 \delta_{ij} \sum_m w_{mi}\left(t_1\right)\Bigg($$
$$\int_{t_0}^{t_1} \mathrm{d}t_3\, G_{mr}^{(0),>}\left(t_1,t_3\right) \sum_{rs} w_{rs}\left(t_3\right) G_{rs}^{(0),<}\left(t_3,t_3\right) G_{sm}^{(0),<}\left(t_3,t_1\right)$$
$$+ \int_{t_1}^{t_0} \mathrm{d}t_3\, G_{mr}^{(0),<}\left(t_1,t_3\right) \sum_{rs} w_{rs}\left(t_3\right) G_{rs}^{(0),<}\left(t_3,t_3\right) G_{sm}^{(0),>}\left(t_3,t_1\right)\Bigg)\,.$$

Further,

$$\Sigma_{ij}^{(2),\{\mathrm{F},0\},\{1,\{\mathrm{H},0\},0\},\mathrm{diagonal},\delta}\left(t_1,t_2\right) \tag{129}$$
$$= \pm\left(\mathrm{i}\hbar\right)^2 w_{ji}\left(t_1\right)\Bigg($$
$$\int_{t_0}^{t_1} \mathrm{d}t_3\, \sum_p G_{ip}^{(0),>}\left(t_1,t_3\right) \sum_{rs} w_{rp}\left(t_3\right) G_{rr}^{(0),<}\left(t_3,t_3\right) G_{pj}^{(0),<}\left(t_3,t_1\right)$$
$$+ \int_{t_1}^{t_0} \mathrm{d}t_3\, \sum_p G_{ip}^{(0),<}\left(t_1,t_3\right) \sum_{rs} w_{rp}\left(t_3\right) G_{rr}^{(0),<}\left(t_3,t_3\right) G_{pj}^{(0),>}\left(t_3,t_1\right)\Bigg)$$

and

$$\Sigma_{ij}^{(2),\{\mathrm{F},0\},\{1,\{\mathrm{F},0\},0\},\delta}\left(t_1\right) \tag{130}$$
$$= \left(\mathrm{i}\hbar\right)^2 \sum_n w_{ji}\left(t_1\right)\Bigg($$
$$\int_{t_0}^{t_1} \mathrm{d}t_3\, G_{ir}^{(0),<}\left(t_1,t_3\right) \sum_{rs} w_{rs}\left(t_3\right) G_{rs}^{(0),<}\left(t_3,t_3\right) G_{sj}^{(0),>}\left(t_3,t_1\right)$$
$$+ \int_{t_1}^{t_0} \mathrm{d}t_3\, G_{ir}^{(0),>}\left(t_1,t_3\right) \sum_{rs} w_{rs}\left(t_3\right) G_{rs}^{(0),<}\left(t_3,t_3\right) G_{sj}^{(0),<}\left(t_3,t_1\right)\Bigg)\,.$$

2.3. Hubbard basis

In the Hubbard basis, the non-selfconsistent contributions attain the form

$$\Sigma_{i\alpha j\alpha}^{(2),(2),2,0,0,\mathrm{Hubbard},\mathfrak{b}}\left(z_1,z_2\right) \tag{131}$$
$$\equiv \Sigma_{i\alpha j\alpha}^{(2),2,0,\mathrm{Hubbard},\mathfrak{b}}\left(z_1,z_2\right)\left(G \rightarrow G^{(0)}\right)$$

and

$$\Sigma_{i\alpha j\alpha}^{(2),(2),1,0,1,\mathrm{Hubbard},\mathfrak{b}}\left(z_1,z_2\right) \tag{132}$$
$$\equiv \Sigma_{i\alpha j\alpha}^{(2),1,1,\mathrm{Hubbard},\mathfrak{b}}\left(z_1,z_2\right)\left(G \rightarrow G^{(0)}\right)\,,$$

as well as

$$\Sigma^{(2),\{\mathrm{H},0\},\{1,\{\mathrm{H},0\},0\},\mathrm{Hubbard},\mathfrak{b}}_{i\alpha j\alpha}\Big(z_1, z_2\Big) \tag{133}$$
$$= \Big(\mathrm{i}\hbar\Big)^2 \delta_{\mathcal{C}}\Big(z_1, z_2\Big)\delta_{ij} \sum_{\epsilon} U\Big(z_1\Big)$$
$$\int_{\mathcal{C}} \mathrm{d}z_3 \sum_p G^{(0)}_{i\epsilon p\epsilon}\Big(z_1, z_3\Big) G^{(0)}_{p\epsilon i\epsilon}\Big(z_3, z_{1^+}\Big) \sum_{\zeta} U\Big(z_3\Big) G^{(0)}_{p\zeta p\zeta}\Big(z_3, z_{3^+}\Big)$$

and

$$\Sigma^{(2),\{\mathrm{H},0\},\{1,\{\mathrm{F},0\},0\},\mathrm{Hubbard},\mathfrak{b}}_{i\alpha j\alpha}\Big(z_1, z_2\Big) \tag{134}$$
$$= \Big(\mathrm{i}\hbar\Big)^2 \delta_{\mathcal{C}}\Big(z_1, z_2\Big)\delta_{ij} \sum_{\epsilon} U\Big(z_1\Big)$$
$$\int_{\mathcal{C}} \mathrm{d}z_3 \sum_p G^{(0)}_{i\epsilon p\epsilon}\Big(z_1, z_3\Big) U\Big(z_3\Big) G^{(0)}_{p\epsilon p\epsilon}\Big(z_3, z_{3^+}\Big) G^{(0)}_{p\epsilon i\epsilon}\Big(z_3, z_{1^+}\Big) .$$

Further,

$$\Sigma^{(2),\{\mathrm{F},0\},\{1,\{\mathrm{H},0\},0\},\mathrm{Hubbard},\mathfrak{b}}_{i\alpha j\alpha}\Big(z_1, z_2\Big) \tag{135}$$
$$= \Big(\mathrm{i}\hbar\Big)^2 \delta_{\mathcal{C}}\Big(z_1, z_2\Big)\delta_{ij} U\Big(z_1\Big)$$
$$\int_{\mathcal{C}} \mathrm{d}z_3 \sum_p G^{(0)}_{i\alpha p\alpha}\Big(z_1, z_3\Big) U\Big(z_3\Big) \sum_{\epsilon} G^{(0)}_{p\epsilon p\epsilon}\Big(z_3, z_{3^+}\Big) G^{(0)}_{p\alpha i\alpha}\Big(z_3, z_{1^+}\Big)$$

and

$$\Sigma^{(2),\{\mathrm{F},0\},\{1,\{\mathrm{F},0\},0\},\mathrm{Hubbard},\mathfrak{b}}_{i\alpha j\alpha}\Big(z_1, z_2\Big) \tag{136}$$
$$= \Big(\mathrm{i}\hbar\Big)^2 \delta_{\mathcal{C}}\Big(z_1, z_2\Big)\delta_{ij} U\Big(z_1\Big)$$
$$\int_{\mathcal{C}} \mathrm{d}z_3 \sum_p G^{(0)}_{i\alpha p\alpha}\Big(z_1, z_3\Big) U\Big(z_1\Big) G^{(0)}_{p\alpha p\alpha}\Big(z_1, z_{1^+}\Big) G^{(0)}_{p\alpha i\alpha}\Big(z_3, z_{1^+}\Big) ,$$

for bosons, and

$$\Sigma^{(2),(2),2,0,0,\mathrm{Hubbard},\mathfrak{f}}_{i\alpha j\alpha}\Big(z_1, z_2\Big) \tag{137}$$
$$\equiv \Sigma^{(2),2,0,\mathrm{Hubbard},\mathfrak{f}}_{i\alpha j\alpha}\Big(z_1, z_2\Big)\Big(G \to G^{(0)}\Big)$$

and

$$\Sigma^{(2),(2),1,0,1,\mathrm{Hubbard},\mathfrak{f}}_{i\alpha j\alpha}\Big(z_1, z_2\Big) \tag{138}$$
$$\equiv \Sigma^{(2),1,1,\mathrm{Hubbard},\mathfrak{f}}_{i\alpha j\alpha}\Big(z_1, z_2\Big)\Big(G \to G^{(0)}\Big) ,$$

as well as

$$\Sigma^{(2),\{\mathrm{H},0\},\{1,\{\mathrm{H},0\},0\},\mathrm{Hubbard},\mathfrak{f}}_{i\alpha j\alpha}\Big(z_1,z_2\Big) \tag{139}$$
$$= \Big(\mathrm{i}\hbar\Big)^2 \delta_{\mathcal{C}}\Big(z_1,z_2\Big)\delta_{ij}\sum_{\epsilon\neq\alpha} U\Big(z_1\Big)$$
$$\int_{\mathcal{C}} \mathrm{d}z_3 \sum_p G^{(0)}_{i\epsilon p\epsilon}\Big(z_1,z_3\Big)G^{(0)}_{p\epsilon i\epsilon}\Big(z_3,z_{1^+}\Big)\sum_{\zeta\neq\epsilon} U\Big(z_3\Big)G^{(0)}_{p\zeta p\zeta}\Big(z_3,z_{3^+}\Big)\,,$$
$$\Sigma^{(2),\{\mathrm{H},0\},\{1,\{\mathrm{F},0\},0\},\mathrm{Hubbard},\mathfrak{f}}_{i\alpha j\alpha}\Big(z_1,z_2\Big) \equiv 0 \tag{140}$$

and

$$\Sigma^{(2),\{\mathrm{F},0\},\{1,\{\mathrm{H},0\},0\},\mathrm{Hubbard},\mathfrak{f}}_{i\alpha j\alpha}\Big(z_1,z_2\Big) \equiv 0\,, \tag{141}$$
$$\Sigma^{(2),\{\mathrm{F},0\},\{1,\{\mathrm{F},0\},0\},\mathrm{Hubbard},\mathfrak{f}}_{i\alpha j\alpha}\Big(z_1,z_2\Big) \equiv 0\,, \tag{142}$$

for fermions. The components read

$$\Sigma^{(2),(2),2,0,0,\mathrm{Hubbard},\mathfrak{b},\gtrless}_{i\alpha j\alpha}\Big(t_1,t_2\Big) \tag{143}$$
$$\equiv \Sigma^{(2),2,0,\mathrm{Hubbard},\mathfrak{b},\gtrless}_{i\alpha j\alpha}\Big(t_1,t_2\Big)\Big(G\to G^{(0)}\Big)\,,$$

$$\Sigma^{(2),(2),1,0,1,\mathrm{Hubbard},\mathfrak{b},\gtrless}_{i\alpha j\alpha}\Big(t_1,t_2\Big) \tag{144}$$
$$\equiv \Sigma^{(2),1,1,\mathrm{Hubbard},\mathfrak{b},\gtrless}_{i\alpha j\alpha}\Big(t_1,t_2\Big)\Big(G\to G^{(0)}\Big)\,,$$

$$\Sigma^{(2),(2),2,0,0,\mathrm{Hubbard},\mathfrak{f},\gtrless}_{i\alpha j\alpha}\Big(t_1,t_2\Big) \tag{145}$$
$$\equiv \Sigma^{(2),2,0,\mathrm{Hubbard},\mathfrak{f},\gtrless}_{i\alpha j\alpha}\Big(t_1,t_2\Big)\Big(G\to G^{(0)}\Big)\,,$$

$$\Sigma^{(2),(2),1,0,1,\mathrm{Hubbard},\mathfrak{f},\gtrless}_{i\alpha j\alpha}\Big(t_1,t_2\Big) \tag{146}$$
$$\equiv \Sigma^{(2),1,1,\mathrm{Hubbard},\mathfrak{f},\gtrless}_{i\alpha j\alpha}\Big(t_1,t_2\Big)\Big(G\to G^{(0)}\Big)\,,$$

as well as

$$\Sigma^{(2),\{\mathrm{H},0\},\{1,\{\mathrm{H},0\},0\},\mathrm{Hubbard},\mathfrak{b},\delta}_{i\alpha j\alpha}\Big(t_1,t_2\Big) \tag{147}$$
$$= \Big(\mathrm{i}\hbar\Big)^2 \delta_{ij}\sum_{\epsilon} U\Big(t_1\Big)\Bigg($$
$$\int_{t_0}^{t_1} \mathrm{d}t_3 \sum_p G^{(0),>}_{i\epsilon p\epsilon}\Big(t_1,t_3\Big)G^{(0),<}_{p\epsilon i\epsilon}\Big(t_3,t_1\Big)\sum_{\zeta} U\Big(t_3\Big)G^{(0),<}_{p\zeta p\zeta}\Big(t_3,t_3\Big)$$
$$+\int_{t_1}^{t_0} \mathrm{d}t_3 \sum_p G^{(0),<}_{i\epsilon p\epsilon}\Big(t_1,t_3\Big)G^{(0),>}_{p\epsilon i\epsilon}\Big(t_3,t_1\Big)\sum_{\zeta} U\Big(t_3\Big)G^{(0),<}_{p\zeta p\zeta}\Big(t_3,t_3\Big)\Bigg)$$

and

$$\Sigma^{(2),\{\mathrm{H},0\},\{1,\{\mathrm{F},0\},0\},\mathrm{Hubbard},\mathfrak{b},\delta}_{i\alpha j\alpha}\left(t_1,t_2\right) \tag{148}$$
$$= \left(\mathrm{i}\hbar\right)^2\delta_{ij}\sum_{\epsilon}U\left(t_1\right)\Bigg(\int_{t_0}^{t_1}\mathrm{d}t_3\sum_p G^{(0),>}_{i\epsilon p\epsilon}\left(t_1,t_3\right)U\left(t_3\right)G^{(0),<}_{p\epsilon p\epsilon}\left(t_3,t_3\right)G^{(0),<}_{p\epsilon i\alpha}\left(t_3,t_1\right)$$
$$+\int_{t_1}^{t_0}\mathrm{d}t_3\sum_p G^{(0),<}_{i\epsilon p\epsilon}\left(t_1,t_3\right)U\left(t_3\right)G^{(0),<}_{p\epsilon p\epsilon}\left(t_3,t_3\right)G^{(0),>}_{p\epsilon i\epsilon}\left(t_3,t_1\right)\Bigg)\,.$$

Further,

$$\Sigma^{(2),\{\mathrm{F},0\},\{1,\{\mathrm{H},0\},0\},\mathrm{Hubbard},\mathfrak{b},\delta}_{i\alpha j\alpha}\left(t_1,t_2\right) \tag{149}$$
$$= \left(\mathrm{i}\hbar\right)^2\delta_{ij}U\left(z_1\right)\Bigg(\int_{t_0}^{t_1}\mathrm{d}t_3\sum_p G^{(0),>}_{i\alpha p\alpha}\left(t_1,t_3\right)U\left(t_3\right)\sum_{\epsilon}G^{(0),<}_{p\epsilon p\epsilon}\left(t_3,t_3\right)G^{(0),<}_{p\alpha i\alpha}\left(t_3,t_1\right)$$
$$+\int_{t_1}^{t_0}\mathrm{d}t_3\sum_p G^{(0),<}_{i\alpha p\alpha}\left(t_1,t_3\right)U\left(t_3\right)\sum_{\epsilon}G^{(0),<}_{p\epsilon p\epsilon}\left(t_3,t_3\right)G^{(0),>}_{p\alpha i\alpha}\left(t_3,t_1\right)\Bigg)$$

and

$$\Sigma^{(2),\{\mathrm{F},0\},\{1,\{\mathrm{F},0\},0\},\mathrm{Hubbard},\mathfrak{b},\delta}_{i\alpha j\alpha}\left(t_1,t_2\right) \tag{150}$$
$$= \left(\mathrm{i}\hbar\right)^2\delta_{ij}U\left(t_1\right)\Bigg(\int_{t_0}^{t_1}\mathrm{d}t_3\sum_p G^{(0),>}_{i\alpha p\alpha}\left(t_1,t_3\right)U\left(t_3\right)G^{(0),<}_{p\alpha p\alpha}\left(t_3,t_3\right)G^{(0),<}_{p\alpha i\alpha}\left(t_3,t_1\right)$$
$$+\int_{t_1}^{t_0}\mathrm{d}t_3\sum_p G^{(0),<}_{i\alpha p\alpha}\left(t_1,t_3\right)U\left(t_3\right)G^{(0),<}_{p\alpha p\alpha}\left(t_3,t_3\right)G^{(0),>}_{p\alpha i\alpha}\left(t_3,t_1\right)\Bigg)$$

and

$$\Sigma^{(2),\{\mathrm{H},0\},\{1,\{\mathrm{H},0\},0\},\mathrm{Hubbard},\mathfrak{f},\delta}_{i\alpha j\alpha}\left(t_1,t_2\right) \tag{151}$$
$$= \left(\mathrm{i}\hbar\right)^2\delta_{ij}\sum_{\epsilon\neq\alpha}U\left(t_1\right)\Bigg(\int_{t_0}^{t_1}\mathrm{d}t_3\sum_p G^{(0),>}_{i\epsilon p\epsilon}\left(t_1,t_3\right)G^{(0),<}_{p\epsilon i\epsilon}\left(t_3,t_1\right)\sum_{\zeta\neq\epsilon}U\left(t_3\right)G^{(0),<}_{p\zeta p\zeta}\left(t_3,t_3\right)$$
$$+\int_{t_1}^{t_0}\mathrm{d}t_3\sum_p G^{(0),<}_{i\epsilon p\epsilon}\left(t_1,t_3\right)G^{(0),>}_{p\epsilon i\epsilon}\left(t_3,t_1\right)\sum_{\zeta\neq\epsilon}U\left(t_3\right)G^{(0),<}_{p\zeta p\zeta}\left(t_3,t_3\right)\Bigg)\,.$$

2.4. Spin-0 bosons/spin-1/2 fermions

For the specific bosonic and fermionic cases, one has

$$\Sigma_{ij}^{(2),(2),2,0,0,\mathrm{Hubbard},\mathfrak{b},0}\Big(z_1,z_2\Big) \equiv \Sigma_{ij}^{(2),2,0,\mathrm{Hubbard},\mathfrak{b},0}\Big(z_1,z_2\Big)\Big(G \to G^{(0)}\Big) \tag{152}$$

and

$$\Sigma_{ij}^{(2),(2),1,0,1,\mathrm{Hubbard},\mathfrak{b},0}\Big(z_1,z_2\Big) \equiv \Sigma_{ij}^{(2),1,1,\mathrm{Hubbard},\mathfrak{b},0}\Big(z_1,z_2\Big)\Big(G \to G^{(0)}\Big)\,, \tag{153}$$

as well as

$$\begin{aligned} &\Sigma_{ij}^{(2),\{\mathrm{H},0\},\{1,\{\mathrm{H},0\},0\},\mathrm{Hubbard},\mathfrak{b},0}\Big(z_1,z_2\Big) \\ &\quad = \Sigma_{ij}^{(2),\{\mathrm{H},0\},\{1,\{\mathrm{F},0\},0\},\mathrm{Hubbard},\mathfrak{b},0}\Big(z_1,z_2\Big) \\ &\quad = \Sigma_{ij}^{(2),\{\mathrm{F},0\},\{1,\{\mathrm{H},0\},0\},\mathrm{Hubbard},\mathfrak{b},0}\Big(z_1,z_2\Big) \\ &\quad = \Sigma_{ij}^{(2),\{\mathrm{F},0\},\{1,\{\mathrm{F},0\},0\},\mathrm{Hubbard},\mathfrak{b},0}\Big(z_1,z_2\Big) \\ &\quad = \Big(\mathrm{i}\hbar\Big)^2\delta_{\mathcal{C}}\Big(z_1,z_2\Big)\delta_{ij}U\Big(z_1\Big) \\ &\qquad \int_{\mathcal{C}}\mathrm{d}z_3\,\sum_p G_{ip}^{(0)}\Big(z_1,z_3\Big)G_{pi}^{(0)}\Big(z_3,z_{1+}\Big)U\Big(z_3\Big)G_{pp}^{(0)}\Big(z_3,z_{3+}\Big)\,, \end{aligned} \tag{154}$$

for spin-0 bosons, and

$$\Sigma_{i\alpha j\alpha}^{(2),(2),2,0,0,\mathrm{Hubbard},\mathfrak{f},1/2}\Big(z_1,z_2\Big) \equiv \Sigma_{i\alpha j\alpha}^{(2),2,0,\mathrm{Hubbard},\mathfrak{f},1/2}\Big(z_1,z_2\Big)\Big(G \to G^{(0)}\Big) \tag{155}$$

and

$$\Sigma_{i\alpha j\alpha}^{(2),(2),1,0,1,\mathrm{Hubbard},\mathfrak{f},1/2}\Big(z_1,z_2\Big) \equiv \Sigma_{i\alpha j\alpha}^{(2),1,1,\mathrm{Hubbard},\mathfrak{f},1/2}\Big(z_1,z_2\Big)\Big(G \to G^{(0)}\Big)\,, \tag{156}$$

as well as

$$\begin{aligned} &\Sigma_{i\uparrow j\uparrow}^{(2),\{\mathrm{H},0\},\{1,\{\mathrm{H},0\},0\},\mathrm{Hubbard},\mathfrak{f},1/2}\Big(z_1,z_2\Big) \\ &\quad = \Big(\mathrm{i}\hbar\Big)^2\delta_{\mathcal{C}}\Big(z_1,z_2\Big)\delta_{ij}U\Big(z_1\Big) \\ &\qquad \int_{\mathcal{C}}\mathrm{d}z_3\,\sum_p G_{i\downarrow p\downarrow}^{(0)}\Big(z_1,z_3\Big)G_{p\downarrow i\downarrow}^{(0)}\Big(z_3,z_{1+}\Big)U\Big(z_3\Big)G_{p\uparrow p\uparrow}^{(0)}\Big(z_3,z_{3+}\Big) \end{aligned} \tag{157}$$

and

$$\Sigma^{(2),\{\mathrm{H},0\},\{1,\{\mathrm{H},0\},0\},\mathrm{Hubbard},\mathfrak{f},1/2}_{i\downarrow j\downarrow}\left(z_1, z_2\right) \tag{158}$$
$$= \left(\mathrm{i}\hbar\right)^2 \delta_{\mathcal{C}}\left(z_1, z_2\right)\delta_{ij} U\left(z_1\right)$$
$$\int_{\mathcal{C}} \mathrm{d}z_3 \sum_p G^{(0)}_{i\uparrow p\uparrow}\left(z_1, z_3\right) G^{(0)}_{p\uparrow i\uparrow}\left(z_3, z_{1^+}\right) U\left(z_3\right) G^{(0)}_{p\downarrow p\downarrow}\left(z_3, z_{3^+}\right),$$

for spin-1/2 fermions. The corresponding components read

$$\Sigma^{(2),(2),2,0,0,\mathrm{Hubbard},\mathfrak{b},0,\gtrless}_{ij}\left(t_1, t_2\right) \tag{159}$$
$$\equiv \Sigma^{(2),2,0,\mathrm{Hubbard},\mathfrak{b},0,\gtrless}_{ij}\left(t_1, t_2\right)\left(G \to G^{(0)}\right),$$

$$\Sigma^{(2),(2),1,0,1,\mathrm{Hubbard},\mathfrak{b},0,\gtrless}_{ij}\left(t_1, t_2\right) \tag{160}$$
$$\equiv \Sigma^{(2),1,1,\mathrm{Hubbard},\mathfrak{b},0,\gtrless}_{ij}\left(t_1, t_2\right)\left(G \to G^{(0)}\right),$$

$$\Sigma^{(2),(2),2,0,0,\mathrm{Hubbard},\mathfrak{f},1/2,\gtrless}_{i\alpha j\alpha}\left(t_1, t_2\right) \tag{161}$$
$$\equiv \Sigma^{(2),2,0,\mathrm{Hubbard},\mathfrak{f},1/2,\gtrless}_{i\alpha j\alpha}\left(t_1, t_2\right)\left(G \to G^{(0)}\right),$$

$$\Sigma^{(2),(2),1,0,1,\mathrm{Hubbard},\mathfrak{f},1/2,\gtrless}_{i\alpha j\alpha}\left(t_1, t_2\right) \tag{162}$$
$$\equiv \Sigma^{(2),1,1,\mathrm{Hubbard},\mathfrak{f},1/2,\gtrless}_{i\alpha j\alpha}\left(t_1, t_2\right)\left(G \to G^{(0)}\right),$$

as well as

$$\Sigma^{(2),\{\mathrm{H},0\},\{1,\{\mathrm{H},0\},0\},\mathrm{Hubbard},\mathfrak{b},0,\delta}_{ij}\left(t_1, t_2\right) \tag{163}$$
$$= \Sigma^{(2),\{\mathrm{H},0\},\{1,\{\mathrm{F},0\},0\},\mathrm{Hubbard},\mathfrak{b},0,\delta}_{ij}\left(t_1, t_2\right)$$
$$= \Sigma^{(2),\{\mathrm{F},0\},\{1,\{\mathrm{H},0\},0\},\mathrm{Hubbard},\mathfrak{b},0,\delta}_{ij}\left(t_1, t_2\right)$$
$$= \Sigma^{(2),\{\mathrm{F},0\},\{1,\{\mathrm{F},0\},0\},\mathrm{Hubbard},\mathfrak{b},0,\delta}_{ij}\left(t_1, t_2\right)$$
$$= \left(\mathrm{i}\hbar\right)^2 \delta_{ij} U\left(t_1\right)\Bigg($$
$$\int_{t_0}^{t_1} \mathrm{d}t_3 \sum_p G^{(0),>}_{ip}\left(t_1, t_3\right) G^{(0),<}_{pi}\left(t_3, t_1\right) U\left(t_3\right) G^{(0),<}_{pp}\left(t_3, t_3\right)$$
$$+ \int_{t_1}^{t_0} \mathrm{d}t_3 \sum_p G^{(0),<}_{ip}\left(t_1, t_3\right) G^{(0),>}_{pi}\left(t_3, t_1\right) U\left(t_3\right) G^{(0),<}_{pp}\left(t_3, t_3\right)\Bigg),$$

for spin-0 bosons, and

$$\Sigma^{(2),\{\mathrm{H},0\},\{1,\{\mathrm{H},0\},0\},\mathrm{Hubbard},\mathfrak{f},1/2,\delta}_{i\uparrow j\uparrow}\Big(t_1,t_2\Big) \tag{164}$$
$$= \Big(\mathrm{i}\hbar\Big)^2\delta_{ij}U\Big(t_1\Big)\Bigg($$
$$\int_{t_0}^{t_1}\mathrm{d}t_3\,\sum_p G^{(0),>}_{i\downarrow p\downarrow}\Big(t_1,t_3\Big)G^{(0),<}_{p\downarrow i\downarrow}\Big(t_3,t_1\Big)U\Big(t_3\Big)G^{(0),<}_{p\uparrow p\uparrow}\Big(t_3,t_3\Big)$$
$$+\int_{t_1}^{t_0}\mathrm{d}t_3\,\sum_p G^{(0),<}_{i\downarrow p\downarrow}\Big(t_1,t_3\Big)G^{(0),>}_{p\downarrow i\downarrow}\Big(t_3,t_1\Big)U\Big(t_3\Big)G^{(0),<}_{p\uparrow p\uparrow}\Big(t_3,t_3\Big)\Bigg)\,,$$

as well as

$$\Sigma^{(2),\{\mathrm{H},0\},\{1,\{\mathrm{H},0\},0\},\mathrm{Hubbard},\mathfrak{f},1/2,\delta}_{i\downarrow j\downarrow}\Big(t_1,t_2\Big) \tag{165}$$
$$= \Big(\mathrm{i}\hbar\Big)^2\delta_{ij}U\Big(t_1\Big)\Bigg($$
$$\int_{t_0}^{t_1}\mathrm{d}t_3\,\sum_p G^{(0),>}_{i\uparrow p\uparrow}\Big(t_1,t_3\Big)G^{(0),<}_{p\uparrow i\uparrow}\Big(t_3,t_1\Big)U\Big(t_3\Big)G^{(0),<}_{p\downarrow p\downarrow}\Big(t_3,t_3\Big)$$
$$+\int_{t_1}^{t_0}\mathrm{d}t_3\,\sum_p G^{(0),<}_{i\uparrow p\uparrow}\Big(t_1,t_3\Big)G^{(0),>}_{p\uparrow i\uparrow}\Big(t_3,t_1\Big)U\Big(t_3\Big)G^{(0),<}_{p\downarrow p\downarrow}\Big(t_3,t_3\Big)\Bigg)\,,$$

for spin-1/2 fermions.