Real-time Nonequilibrium Green's Functions

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"Physicist history of physics", which is never correct. ... is a sort of conventionalized myth – story that the physicists tell to their students, and those tell to their students, and is not necessarily related to the actual historical development.
R. P. Feynman, "QED: The strange theory of light and matter". Princeton University Press, Princeton, New Jersey 1986

A brief review of early Russian works on the Green's functions applications to many body theory, particularly for nonequilibrium states and processes, is presented. Discussed are some general features and relations of the real-time Nonequilibrium Green's function (NGF) matrices method to some other approaches.

Application of field theoretical concepts and methods, including Green's functions, to many body problems in the Soviet physics in the 50s and 60s of the last century was in the mainstream of the process in world science. Because of the iron curtain and the lack of our personal contacts with western colleagues, it was impossible for us to participate in international meetings and conferences, many of the most important scientific journals became available in our institutes and university libraries only with a several months delay – but this only in Moscow, St. Petersburg (then Leningrad) and a few other major cities. And our main journals with a similar delay became translated into English and distributed through the world by the American Physical Society.

There were several active theoretical groups in the Soviet Union in that period – usually called "schools" – and labeled by the name of their leader: L. D. Landau, I. E. Tamm, N. N. Bogolyubov. The most active in quantum many body theory was the Landau school. I myself belonged to the theoretical group of the P. N. Lebedev Institute, and so to the Tamm school. However, from the very beginning of my scientific activity I participated regularly (like my supervisor V. L. Ginzburg) also in the Landau seminar. Several of my friends were from the closest L. D. Landau circle. So I shared many of their views and preferences. In particular, the fascination by the Feynman-Dyson diagram technique (we used the word diagrams instead of graphs). Certainly, we knew about its equivalence to the alternative approach due to J. Schwinger. Landau himself was perfect, even a virtuoso in using the whole arsenal of mathematical physics methods, however, primarily as a tool for analyzing and solving real physical problems. I suppose he appreciated the diagram technique – he also used this term – as a

highly general, regular and logical way, perfectly adapted, besides all that, for describing real physical phenomena and processes. In the whole collection of volumes of the famous Landau and Lifshits course of Theoretical Physics just that approach is used to present both – quantum field theory and quantum many body theory, as in the books [1] and [2] as well. For me, as probably for many others, the diagram technique is more than just a method for doing calculations. Because of its symbolical but very spectacular presentation in terms of graphs it is more like the way of thinking about physical processes and theoretical approximations.

This rather personal remark is meant to say that the following considerations should not be regarded as a comprehensive review of the subject. Inevitably I shall speak mainly about the activity related to the diagram technique in many body theory, which I knew better. Some alternative approaches, including the one based on Bogolyubov's idea of decoupling the infinite set of equations for GF's of successively increasing particle number by approximating higher order GF's in terms of those of lower order, were reviewed in [3]$^-$[7]. Also, as our Workshop is about the NGF's, only a very short list of some previous works on the many body ground state and the thermodynamic equilibrium state (Matsubara) GF's is presented below (and only Soviet – the Western, I suppose, are known much better). I mainly intend to give an impression of the starting level at the beginning of 1960's.

Probably the first paper on GF's in many body systems in Russian journals was that of Bonch-Bruevich [7], discussing the general concept of GF in the ground state, including the relation of the zeros of $G^{-1}(\mathbf{p}, \varepsilon)$ to the particle energy levels. The systematic studies of the interacting electron-phonon system in metals at $T = 0$ in terms of GF's was started by Migdal and Galitskii [8, 9] resulting, in particular, in the Migdal theorem about the absence of the coupling constant renormalization by the *e-ph* interaction. Galitskii [10] calculated also the energy spectrum and ground state energy of a low density degenerate Fermi gas with short range interaction. Essential was the work of Belyaev [11] who has modified the standard diagram technique in order to describe degenerate systems of bosons. Here for the first time "anomalous" GF's were introduced, accounting for the macroscopic coherency in the system. Similar pair coherency functions for Fermi systems, introduced by Gor'kov [12], became crucial for describing superconductivity within the diagram technique. Using this Gor'kov's technique Abrikosov, Gor'kov, and Khalatnikov were able to develop a theory for the majority of superconductivity related phenomena, including the electromagnetic response [1, 13]. Important for the whole quantum many body theory is the result of Landau himself about the analytical properties of GF's in the complex frequency plane, including spectral representations, valid for any system in thermodynamic equilibrium [14]. He has applied also the diagram technique to justify his Fermi liquid theory and clarify the microscopic nature of its basic notions, including the quasiparticle scattering amplitude [15].

In the following year (1959) the breakthrough occured in the theory of Matsubara's (temperature) GF's. Abrikosov, Gor'kov, and Dzyaloshinskii [16] and Fradkin [17] derived the periodic boundary conditions and were able to introduce the Fourier representation in terms of discrete frequencies, which has made the diagram technique on the imaginary time (temperature) axis an effective tool for calculating any thermodynamic equilibrium parameter of many body systems. These authors have also pointed out the possibility of finding the real time GF's

under thermodynamic equilibrium conditions by means of analytical continuation from the discrete set of Matsubara frequencies to the real frequency axis. In all the above mentioned papers mainly the Feynman causal function $G_c(x, x')$ was discussed, as the standard diagram technique exists just for this function. In the same year Bogolyubov and Tyablikov [18] discussed the possibility of describing the many body system in terms of retarded $G^{(r)}(x, x')$ and advanced $G^{(a)}(x, x')$ functions, using the infinite hierarchy of coupled equations for the GF's with increasing number of particles.

In the beginning of the 1960's the interest in the field shifted to kinetic (transport) phenomena. Initially the focus was on linear response. In classical many body theory kinetics is described usually in the framework of Boltzmann's equation for the particle distribution function in the phase space. In quantum theory the description is based on evolution of the density matrix according to the Schrödinger equation. The closest quantum analogue to the distribution function is the density matrix in the so called Wigner representation. Under quasiclassical conditions the Wigner density matrix can be reduced to the classical distribution function and, corresponding quantum equation, to the usual Boltzmann equation. This is the standard way of introducing quantum corrections to the Boltzmann equation (see e.g. [19]). Real-time GF's are a kind of generalization of the density matrix, reducing to the latter at coinciding time arguments. So, already in [9] the relation of the two-particle GF and the Bethe-Salpeter equation to the linearized (weak field) Boltzmann equation was indicated. The next important step in that direction was done by Konstantinov and Perel' [20]. Expressing the linear correction to the single-particle density matrix in terms of the retarded density-density correlator and introducing for its calculation some special version of the diagram technique on the complex time contour they were able to derive the generalized Boltzmann equation, capable in principle to account for quantum corrections up to any desired order in the interaction. Another approach to the same linear response problem was based on the analytic continuation of Matsubara's GF's onto the real frequency axis as suggested in [16, 17]. Formally the response function is the retarded self-energy of the field propagator in the medium. Eliashberg [21, 22] under quasiclassical conditions – external field, slowly varying in space and time – analyzed the analytical structure of the two particle scattering vertex in the complex frequency plane and so was able to continue the Bethe-Salpeter type equation to real frequencies, resulting in the generalized quantum Boltzmann equation for the Fermi liquid. Later Gor'kov and Eliashberg [23, 24] extended that method to the nonlinear regime for the particular case of superconductors in strong space-time dependent fields. Dzyaloshinskii [25] has found a regular way of continuing each Matsubara-Feynman graph to the real frequency in terms of spectral functions. Graphs become different depending on the time sequence ordering of different interaction vertices, resulting in an exponential increase of the number of graphs in higher orders of the perturbation theory. Generally the linear response problem was solved by these works.

My interest in the NGFs problem started in the very beginning of 1964 and was motivated purely aesthetically. The Feynman graph technique seemed to me so natural and logical that it was hard to believe that its applicability is restricted to an important but very special class of states – ground state or thermodynamic equilibrium. So the program was most simple and straightforward: to follow step by step the original Feynman-Dyson derivation, checking at

what step it becomes invalid for an arbitrary state and then trying to overcome the arising difficulties, staying as close as possible to the original formulation. The first evident difference is that an arbitrary state of a many-body system is described in terms of the density matrix $\hat{\rho}$ while it was a pure ground state (vacuum) $|0\rangle$ in the original procedure. However the time evolution for both is defined by the Schrödinger equation, which for the density matrix is

$$i\frac{\partial\hat{\rho}}{\partial t} = \left[\hat{H},\hat{\rho}\right], \qquad \hat{H} = \hat{H}_0(t) + \hat{H}_{\rm int}.$$

Unlike the original case, $\hat{H}_0$ may be time-dependent because of the presence of external fields. That, however, did not create any essential difficulties. Following the usual procedure of adiabatic switching of the interaction (not external fields, which are switched on realistically, not adiabatically!), in the interaction representation one gets

$$\hat{\rho}(t) = \hat{S}(t,-\infty)\hat{\rho}(-\infty)\hat{S}^\dagger(t,-\infty)\,,$$

instead of the usual $\Psi(t) = \hat{S}(t,-\infty)\Psi_0$, with the standard definition of the S-matrix

$$\begin{aligned} \hat{S}(t,t') &= \hat{T}\exp\left\{-i\int\limits_{t'}^{t}\hat{H}_{\rm int}(t_1)dt_1\right\} \\ \text{and}\quad \hat{S}^\dagger(t,t') &= \hat{T}\exp\left\{-i\int\limits_{t}^{t'}\hat{H}_{\rm int}(t_1)dt_1\right\}. \end{aligned}$$

Here, $\hat{T}$ is the usual time ordering operator along the integration path in the direction of integration, i.e. from the lower to the upper limit. The average value of any physical quantity operator $\hat{L}_0(t)$ (in the interaction representation) is then

$$\begin{aligned} \bar{L}(t) &= {\rm Tr}\left\{\hat{L}_0(t)\hat{\rho}(t)\right\} \\ &= {\rm Tr}\left\{\hat{S}^\dagger(t,-\infty)\hat{L}_0(t)\hat{S}(t,-\infty)\hat{\rho}(-\infty)\right\}, \end{aligned} \tag{1}$$

(cyclic permutation under the Tr symbol is used). This formula means the transition to the Heisenberg representation of time-dependent operators averaged over a time-independent density matrix of non-interacting fields (because of adiabatic switching of the interaction). In the absence of interaction and external fields the density matrix was taken to be $\hat{\rho}(-\infty) = \hat{\rho}_{0T}$ – corresponding to the thermodynamic equilibrium for free particles.

At the next step the original procedure breaks down. This step for the vacuum (ground) state usually is substituting $\hat{S}(\infty,t)$ instead of $\hat{S}^\dagger(t,-\infty)$, justified by the so called "vacuum stability condition", which reads: under adiabatic transformation the non-degenerate ground state can transform only into itself, possibly multiplied by an (unessential) phase factor. Then

$$\begin{aligned} \hat{S}^\dagger(t,-\infty) = \hat{S}(-\infty,t) &= \hat{S}(-\infty,t)\hat{S}(t,\infty)\hat{S}(\infty,t) \\ &= \hat{S}(-\infty,\infty)\hat{S}(\infty,t). \end{aligned}$$

Acting on the vacuum, the first factor on the r.h.s. of this relation is exactly the inverse of that phase factor – a c-number $\langle S\rangle_0^{-1} = \left\langle 0 \left| \hat{S}(\infty,-\infty) \right| 0 \right\rangle^{-1}$. So the well-known formula appears

$$\bar{L}(t) = \langle S\rangle_0^{-1} \cdot \left\langle 0 \left| \hat{T}\left(\hat{S}(\infty,-\infty)\cdot \hat{L}_0(t)\right) \right| 0 \right\rangle .$$

However, for an arbitrary non-equilibrium state created by an external (possibly time-dependent) field, the stability condition cannot be generally valid. So one is forced to proceed with the unchanged formula (1). But then the contour ordering comes automatically. Accounting for the opposite time ordering in $\hat{S}$ and $\hat{S}^\dagger$ the formula (1) can be written as

$$\bar{L}(t) = \mathrm{Tr}\left\{ \hat{T}_C \left(\hat{S}_C \cdot \hat{L}_0(t)\right) \cdot \rho_{0T} \right\}, \tag{2}$$

where, C is the contour propagating from $-\infty$ to time t and then back to $-\infty$, $\hat{T}_C$ is the ordering operator and $\hat{S}_C$ – the S-matrix along this contour. If not under the time ordering symbol together with some other operator $\hat{L}_0(t)$, then $\hat{S}_C \equiv \hat{1}$, which means identical absence of all vacuum loops in this technique[1]. In order to extend all the integrals over the whole time axis, one can insert into (2) one more factor – the operator identically equal to unity – $\hat{S}(t,\infty)\hat{S}(\infty,t)$. It does not change anything. However, the contour C propagates now from $-\infty$ to ∞ and back to $-\infty$, which is much more convenient. It should be noted that both branches of the contour propagate along the real time axis. Any references to the complex time plane were eliminated from this consideration because of usually non-analytical time dependence of external fields (switching). Formulas similar to (2) hold for averaged products of any number of operators, including field operators related, in general, to different times. These are real-time NGF's. Depending on the positions of the times on different branches of the contour they correspond to different time ordering of operators, as times on the reverse branch are oppositely ordered and are always "later" than any time on the direct branch; e.g. for the single-particle GF four different functions exist:

$$\begin{aligned}
G(t_+,t'_+) &= -i\left\langle \hat{T}\left(\psi_0(t),\psi_0^\dagger(t')\right)\right\rangle_{0T} && (3)\\
G(t_+,t'_-) &= -i\left\langle \psi_0^\dagger(t'),\psi_0(t)\right\rangle_{0T} \\
G(t_-,t'_+) &= -i\left\langle \psi_0(t),\psi_0^\dagger(t')\right\rangle_{0T} && (4)\\
G(t_-,t'_-) &= -i\left\langle \hat{\tilde{T}}\left(\psi_0(t),\psi_0^\dagger(t')\right)\right\rangle_{0T} .
\end{aligned}$$

Here $\hat{\tilde{T}}$ is the reverse time ordering operator, and $\langle ...\rangle_{0T}$ denotes averaging over $\hat{\rho}_{0T}$. Together these four functions compose into the single contour ordered GF $G_C(x,x')$.

Now, evidently, the usual diagram technique follows from perturbative representation of (2) and all Feynman rules are valid with only one exception: GF's

[1]Strictly speaking the contour is optional in this derivation. As Pitaevskii has shown, presenting this technique in the volume "Physical Kinetics" of the Landau and Lifshits course on Theoretical Physics, one can get all the same results, considering a perturbation series directly in formula (1) and taking into account opposite ordering in S and $S^\dagger$. Then these two factors play the role of two contour branches. The contour, however, seems more spectacular.

are defined along the whole contour and all time integrals become extended along the whole contour. We can consider the $\pm$ indices of the contour branches as matrix indices. Then the 4 GF's (3) are components of a 2×2 matrix, and multiplication of those matrices accounts for summing up contributions of different contour branches. All time integrals extend only along one real time axis from $-\infty$ to ∞. However, those corresponding to the matrix index "–", should be taken with a "–" sign to account for the reverse direction of integration. Now the difference to the usual Feynman diagram technique is that to each line of the graph corresponds the GF's matrix (3) and to connect 3 or 4 such matrices in each interaction point, the elementary vertex γ^k_{ij} or γ^{kl}_{ij} is introduced being equal to $+1$, if all indices are $+$, -1, if all indices are $-$, and 0 otherwise. Similar contour ordered GF matrices were used also by J. Schwinger in his earlier paper [26] about the Brownian motion of the harmonic quantum oscillator driven by two external forces – one regular (arbitrary function of time) and another, stochastic, defined in terms of its correlators ("thermostat").

Starting from that point the contour may be forgotten: the whole information, which it carried, is now accounted for by the structure of GF matrices. Only one real physical time remains, propagating from $-\infty$ to ∞. Moreover, matrices themselves may be transformed by any canonical transformation, resulting in another equivalent description of the same system. Then the matrix indices cease to be related to contour branches. Therefore instead of $\pm$ indices usual 1,2 should be used. The canonical transformation freedom can be used to reduce the number of acting GFs, as only two of them are linearly independent. In particular, the simple transformation was found transforming (3) into the usually called "triangular" representation

$$\hat{G}_0(x,x') = \begin{pmatrix} 0 & G_0^{(a)}(x,x') \\ G_0^{(r)}(x,x') & F_0(x,x') \end{pmatrix}.$$

Here, the retarded $G_0^{(r)}(x,x')$ and advanced $G_0^{(a)}(x,x')$ GF's are defined as

$$\begin{aligned} G_0^{(r)}(x,x') &= -i\,\Theta(t-t') \left\langle \left[\psi_0(x), \psi_0^\dagger(x')\right]_\mp \right\rangle_{0T} \\ &= \left[G_0^{(a)}(x',x)\right]^* \end{aligned}$$

and

$$F_0(x,x') = -i \left\langle \left[\psi_0(x), \psi_0^\dagger(x')\right]_\pm \right\rangle_{0T},$$

with the upper sign for bosons and the lower for fermions. In these formulae, $\psi_0(x)$ denotes the field operators in the interaction representation, which obey

$$\left[\psi_0(x), \psi_0^\dagger(x')\right]_\pm = \psi_0(x)\psi_0^\dagger(x') \pm \psi_0^\dagger(x')\psi_0(x).$$

The triangular representation has a few evident advantages. It is minimal – the number of nonzero matrix elements cannot be reduced further by *canonical* transformation.

1. It is explicitly time symmetric (like Feynman's $G_{\rm causal}$ in vacuum) – retarded and advanced functions enter symmetrically. This is despite the

existence of the time arrow, which is accounted for by the relative positions of the elements of the GF matrix.

2. It is symmetric in emission and absorption processes. For fermions it is explicitly "charge symmetric", i.e. (anti)symmetric in electrons and holes – $iF_0(\mathbf{p}, t' = t) = (1 - 2n_{\mathbf{p}})$ is positive for empty states and negative for occupied.

3. The functions $G^{(r,a)}(x, x')$ satisfy the universal initial condition

$$G^{(r,a)}(x, x')|_{t'=t} = \mp i\delta(\mathbf{r} - \mathbf{r}')\,,$$

even after complete renormalization, then they are defined in terms of Heisenberg operators. The renormalized function F at coincident time arguments reduces to the single-particle density matrix

$$iF(x, x')|_{t'=t} = \delta(\mathbf{r} - \mathbf{r}') \pm 2\rho(\mathbf{r}, \mathbf{r}'; t).$$

In that sense one can say, that $F(x, x')$ is the generalized distribution function while $G^{(r,a)}(x, x')$ describe essentially the renormalized particle dynamics. After renormalization the GF matrix is defined by the matrix Dyson equation

$$\begin{aligned}\hat{G}(x, x') &= \hat{G}_0(x, x') + \\ &+ \int\int \hat{G}_0(x, y)\hat{\Sigma}(y, y')\hat{G}(y', x')dydy'\,,\end{aligned}$$

which in triangular representation reduces to two equations

$$\begin{aligned}G^{(r)}(x, x') &= G_0^{(r)}(x, x') + \\ &+ \int\int G_0^{(r)}(x, y)\Sigma_r(y, y')G^{(r)}(y', x')dydy'\end{aligned}$$

and

$$F(x, x') = \int\int G^{(r)}(x, y)\Omega(y, y')G^{(a)}(y', x')dydy'.$$

Note that the self energy matrix $\hat{\Sigma}$ is defined in terms of GF matrices as the sum of all exactly the same graphs as in the vacuum field theory or the ground state many-body theory. In the triangular representation

$$\hat{\Sigma} = \begin{pmatrix} \Omega & \Sigma_r \\ \Sigma_a & 0 \end{pmatrix}.$$

Those were the main results in [27], published in 1964.

In the year 1999, G. Baym in his talk at the opening session of the Conference "Kadanoff-Baym Equations – Progress and Perspectives for Many-body Physics" in my absence mentioned that paper. "The method" (round trip contour. *L.K.*) "was then used by Leonid Keldysh in the Soviet Union, described first in his 1964 paper [27]. Our book was translated into Russian in the same

year [28] but Keldysh did not refer to it..."[2]. Evidently, such a sentence is intended to create the impression that some method and may be also results of [27] were adopted from [28]. This statement, extremely unfriendly, being published in [29] needs a reply, especially now, after 38 years, when hardly anybody would check carefully, what was really written in so old papers. First of all, on the last page of the Russian edition of [28] among other typographical information two dates are indicated: "submitted for production 04.07.1964" and "signed for printing 05.10.1964" (still not printed). The paper [27] was published in the October 1964 issue of JETP and was submitted in April 1964. It is hard to imagine how I could refer to this translated version. Sure, however, all that about that Russian edition does not matter. The original version of [28] was published two years earlier and, if it in fact contained all or important parts of the results of [27], then [27] would be deprived of any significance, no matter have I seen [28] or not. Therefore, one should look what is the overlap, if any, of [27] and [28]. That corresponds more closely to the subject of our Workshop – about NGF's. So about the method. I was following, as explained above, the standard Feynman–Dyson method with perturbative expansion, S-matrix, time ordering etc. The authors of [28] used directly Heisenberg's equations of motion, which the authors themselves oppose to "... an alternative scheme, based upon an expansion of G in a power series..." (Ref. [28] p. 191, and further about perturbative series, but *only* equilibrium Matsubara's.). More important, however, is that their method is based completely on the analytic continuation of Matsubara's equations from the imaginary time axis to the real one, while in [27] all the derivation is done on the real time axis in order to get results applicable to experimentally realistic external fields, which are always exactly zero before some switch-on time. So it was impossible for me to use any detail of the method of [28]. However, maybe speaking about "the method" G. Baym meant contour ordering and 2×2 GF matrices. In that case the story becomes even more amusing. There is no contour ordering, no 2×2 GF matrices and even no reference (!) to Schwinger's paper [26] or to his 1960 lectures in the book [28], neither in the text nor in the reference list. In the year 1999, G. Baym considers Schwinger's idea about GFs contour ordering[3] as "The crucial ingredient in the

[2]To be correct that whole text (pp. 28-29) is reproduced below, except for a few lines about J. Schwinger's style:

"A crucial ingredient in the derivation of Boltzmann equations was the use of Green's functions defined on the round-trip contour along the real axis. The method was invented by Schwinger and presented in his lectures on Brownian motion at the Brandeis summer school in 1960, where I became familiar with it. Although the lectures were unpublished, Schwinger did write up his ideas in his paper *Brownian motion of a Quantum Oscillator*. ....

The round trip technique was also employed in the context of quantum electrodynamics in 1961-62 by Kalayana T. Mahanthappa, a fellow Schwinger graduate student at Harvard and Pradip Bakshi, a slightly later student of Schwinger's. Actually, Robert Mills (of Yang-Mills), while at the University of Birmingham in 1962, wrote but did not publish a lovely set of notes on round-trip Greens functions techniques, which formed the basis for his later book. He refers in these notes to Schwinger's 1961 paper and remarks that, 'The present work, some of which has, I believe, been duplicated independently by Baym and Kadanoff, following the methods of Martin and Schwinger, makes use of the thermodynamic Wick's theorem of Matsubara and Thouless, and others, with the integration contour in the complex time plane distorted to include the real axes.' The method then was used by Leonid Keldysh in the Soviet Union, described first in his 1964 paper. Our book was translated into Russian in the same year, but Keldysh did not refer to it, writing rather, 'Our diagram technique will be close to Mills' technique for equilibrium systems', citing Mills' notes. Schwinger's influence was widely felt."

[3]The crucial step is just the contour ordering of GF's. Not the contour itself , which

derivation of Boltzmann equation...". However, in the year 1962, two years after his listening to Schwinger's lectures and a year after publishing [26], he considered it as not worth mentioning in the book entitled *Quantum Statistical Mechanics*.

The major difference between the Kadanoff-Baym equations [28] and the matrix Dyson equation derived in [27] is the algorithm of self energy calculation in terms of GF's. In [27] it is the standard and absolutely regular Feynman diagram technique with GF matrices. In [28], self energies are defined as two analytic components of Matsubara's self energy in the complex time plane, used, however, on the real time axis. No other regular algorithm of self energy calculation is presented. So, strictly speaking, corresponding to that definition procedure must be the analytic continuation of Matsubara's self energy functions from the imaginary time axis to the real times for solving the Kadanoff-Baym equations and then analytic continuation of the GF's from the real-time axis to imaginary times for calculating self energies. In terms of analytic functions $G^<$ and $G^>$ each of Matsubara's graphs of the n-th order transforms into $\sim n!$ graphs, differing by the time sequence of different interaction vertices. That makes the perturbation theory much more cumbersome, much like the old Schrödinger perturbation theory, strictly speaking non-renormalizable. Probably that is the reason why in [28] self energies are calculated only up to the second order in the interaction. Calculation in this approximation does not contain time integrals – self energies are proportional to GF products and do not need separate analytic continuation. The only attempt to go beyond that approximation and introduce the scattering amplitude instead of the interaction potential (chapter 13) is restricted to the equilibrium case. Sure, now there exists the possibility to calculate self energies directly on the real-time axis – in terms of real-time GF matrices, which calculates the whole $\hat{\Sigma}$-matrix including $\Sigma^<$ and $\Sigma^>$. However, there was nothing like that in the book [28].

To summarize that part of my talk, I believe that the paper [27] solved that problem which was announced in the title of that article: "Diagram Technique for Nonequilibrium Processes", which is applicable also to many-body systems in equilibrium or in the ground state. And nothing of the methods developed in [28] was used.[4]

was used already in Ref. [20]. To map two time arguments of GF each onto one of contour branches and later to arrange them into a 2×2 matrix, at least 4 functions are necessary, as it was done in both [26] and [27]. No matter that only 2 functions are linearly independent – one needs to know which place should be ascribed to each of the linear combinations. That could not be done, and was not done, in [28] operating with only two functions - $G^>$ and $G^<$. The Fig. 5 in [29] , p. 26 illustrates well "the method" of G. Baym. It is announced as "The succession of contours in deriving the generalized Boltzmann equation". Plots a) and b) with all integrals along the Matsubara's imaginary time segment correspond indeed to the procedure, described in [28]. Then in the plot c) the deformation to the "real time round trip contour" is presented. But there is nothing like that in [28]. Any idea about the contour, even the word "contour" is absent. Equations (8-27) – the Kadanoff-Baym equations – contain usual (not contour) integrals along the real time axis. All integrals in the preceding formula are along imaginary Matsubara's segment. Analytic continuation is made without any contour. Performing an analytic continuation does not require the contour. Using the contour does not require the analytic continuation, as it was done in both [26] and [27].

[4]As the only overlap of [27] and [28] one may regard the derivation of the Boltzmann equation. However, it was done for different relaxation mechanisms – phonon scattering in [27] and particle collisions in [28]. Moreover, unlike [28], where generalization of the Boltzmann equation was derived, in [27] it was just an illustration without any pretension for any new result. It was done only to the lowest approximation presented in many textbooks and along

Another old problem, only mentioned in my paper is that of initial conditions, i.e. the description of a many-body system, which at some fixed time t_0 was in some arbitrary fixed state $\hat{\rho}_0$. I shall say only a few words about some early papers in that direction. To my knowledge the first solution was presented by Hall [30]. He has shown that the diagram technique becomes modified by including new elements – all the initial correlators contained in $\hat{\rho}_0$, become build in graphs as independent (multi-particle) vertexes. Hall's technique was essentially developed by Kukharenko and Tikhodeev [31] by a renormalization procedure accounting for the decay of initial correlations and slow time evolution of basic parameters, like distribution functions and its higher self correlators, from their initial values at t_0 to following current time t, and so derived generalized Boltzmann equation and a set of transport equations for fluctuation correlations.

My point of view is that in many, may be the majority, of realistic cases initial conditions result from some evolution or external conditions, which both – prehistory and external conditions – can, or should, be included in the consideration, reducing the problem to the evolution from the far past, which is already forgotten. Indeed, as Fanchenko [32] has shown, the problem of arbitrary initial conditions can be always reduced to the evolution, starting from equilibrium at $t \to -\infty$ by including in the Hamiltonian some perturbation $\hat{H}'$, resulting in the evolution from $\hat{\rho}_{0T}$ at $t \to -\infty$ to $\hat{\rho}_0$ at $t = t_0$. So the problem reduces to the diagram technique in terms of usual GF matrices, with substitution, however, of complicated vertices from $\hat{H}'$ instead of initial correlation blocks of Hall's technique. In physically realistic cases $\hat{H}'$ hardly can be very complicated.

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the well known lines: GF at coincident time arguments – density matrix, Wigner function, imaginary parts of self energies – scattering rates etc. As a generalization of the Boltzmann equation the matrix Dyson equation was called.

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