Annals of Physics 323 (2008) 3158–3174

Contents lists available at ScienceDirect

Annals of Physics

journal homepage: www.elsevier.com/locate/aop

Quantum kinetic theory of trapped particles in a strong electromagnetic field

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ARTICLE INFO

Article history:
Received 3 June 2008
Accepted 4 August 2008
Available online 19 August 2008

Keywords:
Quantum kinetic theory
Effective quantum potentials
Strong electromagnetic fields
Nonequilibrium Green's functions
Confined charges

ABSTRACT

The idea of treating quantum systems by semiclassical representations using effective quantum potentials (forces) has been successfully applied in equilibrium by many authors, see e.g. [D. Bohm, Phys. Rev. 85 (1986) 166 and 180; D.K. Ferry, J.R. Zhou, Phys. Rev. B 48 (1993) 7944; A.V. Filinov, M. Bonitz, W. Ebeling, J. Phys. A 36 (2003) 5957 and references cited therein]. Here, this idea is extended to nonequilibrium quantum systems in an external field. A gauge-invariant quantum kinetic theory for weakly inhomogeneous charged particle systems in a strong electromagnetic field is developed within the framework of nonequilibrium Green's functions. The equation for the spectral density is simplified by introducing a classical (local) form for the kinetics. Nonlocal quantum effects are accounted for in this way by replacing the bare external confinement potential with an effective quantum potential. The equation for this effective potential is identified and solved for weak inhomogeneity in the collisionless limit. The resulting nonequilibrium spectral function is used to determine the density of states and the modification of the Born collision operator in the kinetic equation for the Wigner function due to quantum confinement effects.

1. Introduction

Correlated charged particles in external confinement potentials are of growing interest in many fields of physics. Examples are valence electrons in metal clusters, e.g. [4–6], electrons in quantum

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doi:10.1016/j.aop.2008.08.001

dots, e.g. [7–9], dusty plasmas [10,11] or trapped ultracold atoms or ions [12,13]. Of special relevance is the response of these systems to a time-dependent electromagnetic field including, in particular, high intensity fields generated by novel radiation sources like short-pulse lasers or free electron lasers. Being exposed to such fields, the charged particle ensemble will typically be driven far away from equilibrium. To describe these phenomena, a general nonequilibrium approach is necessary which consistently includes field–matter interaction and many-particle and quantum effects.

Nonequilibrium Green's functions provide a powerful method to treat strong field effects in quantum many-body systems, thereby fully perserving conservation laws [15a,15b]. and have been successfully used by many authors; for an overview cf. [14–16] and references therein. Among the central problems is that the results of many-body approximations (such as gradient expansions) are known to be dependent on the chosen gauge. Therefore, an explicit gauge-invariant theory [17] provides a convenient starting point from which approximations can be systematically derived. This method turned out to be fruitful in many fields including semiconductor quantum transport, e.g. [16], and dense quantum plasmas [19,20] and laser plasmas [21], for a recent overview see [22].

The latter results were limited to spatially homogeneous systems. For the present case of spatially confined systems it is necessary to extend them to the inhomogeneous case. In many situations of trapped particles the confinement potential is varying relatively smoothly and slowly compared to the inter-particle distance or the mean free path, which allows us to directly extend previous works [19,20]. In previous papers [23–25] first results along this path have been presented for valence electrons in metal clusters. The objective here is to give a brief review of this gauge invariant nonequilibrium Green's function representation (Sections 2.1 and 2.2), and a practical application to obtain a quantum kinetic theory of confined particles (both fermions and bosons) in a strong electromagnetic field. The central new feature of this analysis is the inclusion of the quantum effects associated with both the confining potential and external fields via the introduction of a nonequilibrium local quantum potential which generalizes earlier concepts of stationary quantum potentials, see e.g. Ref. [1–3,30,31]. This is described in Section 2.3 and is used to determine the spectral function and spectral density in the collisionless limit as a function of this potential and arbitrary external electromagnetic field. The corresponding self-consistent equation for the quantum potential is solved in Section 3 for a slowly varying (in space and time) confining potential. Explicit forms for the spectral density and density of states are considered to demonstrate the quantum effects.

The kinetic theory for the Wigner distribution in the Born approximation is considered in Section 4. It is noted that the nonlocal quantum effects of the confining potential in this equation lead to a different quantum potential, using an approach similar to that of Section 3 for the spectral density. The details of this potential are not pursued here. Instead, attention is focused on the effects of the spectral density on the collision operator. Coulomb collisions are described by eliminating the two-time correlation function via the generalized Kadanoff–Baym ansatz [18–20]. The resulting collision operator is expressed in terms of the spectral functions. Using the results of Section 3 these are expressed in terms of the quantum potential, showing how the Coulomb collisions are modified by quantum effects associated with this confinement potential.

2. Gauge-invariant quantum kinetic theory of trapped particles in a strong electromagnetic field

In this paper, we consider an ensemble of N trapped correlated quantum particles subject to a strong time-dependent classical electromagnetic field which are described by the Hamiltonian

$$\widehat{H} = \sum_{i=1}^{N} \left\{ \epsilon\left(\frac{\hbar}{i}\nabla_i - \frac{e_a}{c}\mathbf{A}(\mathbf{r}_i,t)\right) + \phi(\mathbf{r}_i,t) + V(\mathbf{r}_i,t) + \sum_{j<i}^{N} \frac{e_a^2}{|\mathbf{r}_i - \mathbf{r}_j|} \right\}. \tag{1}$$

Here a labels the particle species, $\epsilon(\hat{p})$ is the single-particle energy dispersion, e.g. band structure, and V is the confinement potential, e.g. an external trapping potential that can be time dependent.

2.1. Kadanoff–Baym/Keldysh equations

The field theoretical description of quantum particles is based on the bosonic/fermionic creation and annihilation operators $\psi^\dagger$ and ψ which are defined to guarantee the spin statistics theorem

$$\psi(1)\psi(1') \mp \psi(1')\psi(1) = \psi^\dagger(1)\psi^\dagger(1') \mp \psi^\dagger(1')\psi^\dagger(1) = 0,$$
$$\psi(1)\psi^\dagger(1') \mp \psi^\dagger(1')\psi(1) = \delta(1-1'),$$

where $t_1 = t_1'$ has been assumed, and $1 \equiv (\mathbf{r}_1, t_1, s_1^3)$. Below, we will drop the spin index as it is not relevant for our analysis. The nonequilibrium state is described by the two-time correlation functions which are statistical averages (with the initial density operator of the system) of field operator products

$$g^>(1,1') = \frac{1}{\mathrm{i}\hbar}\langle\psi(1)\psi^\dagger(1')\rangle, \qquad g^<(1,1') = \pm\frac{1}{\mathrm{i}\hbar}\langle\psi^\dagger(1')\psi(1)\rangle, \tag{2}$$

where in nonequilibrium, $g^>$ and $g^<$ are independent from each other and contain the complete dynamical and statistical information. In the present case, we will be interested in the one-particle density matrix obtained from the function $g^<$ along the time diagonal according to

$$F_1(\mathbf{r}_1, \mathbf{r}_1', t) = \pm\mathrm{i}\hbar g^<(1,1')|_{t_1=t_1'}, \tag{3}$$

and the spectral function from values across the diagonal in the t_1–t_1'-plane

$$a(1,1') \equiv \mathrm{i}\hbar\{g^>(1,1') - g^<(1,1')\} = \mathrm{i}\hbar\{g^R(1,1') - g^A(1,1')\}, \tag{4}$$

where $g^{\mathrm{R/A}}$ are the retarded and advanced Green's functions

$$g^{\mathrm{R/A}}(1,1') = \pm\Theta[\pm(t_1 - t_1')]\{g^>(1,1') - g^<(1,1')\}. \tag{5}$$

The equations of motion for the two-time correlation functions are the Kadanoff–Baym/Keldysh equations (KBE) which, in the presence of a strong homogeneous electromagnetic field and an inhomogeneous confinement potential V, read [23]

$$\left[\mathrm{i}\hbar\frac{\partial}{\partial t_1} - \epsilon\left(\frac{\hbar}{i}\nabla_1 - \frac{e_a}{c}\mathbf{A}(t_1)\right) - V(1)\right]g^{\gtrless}(1,1') = \int \mathrm{d}\bar{\mathbf{r}}_1\Sigma^{\mathrm{HF}}(1,\bar{\mathbf{r}}_1 t_1)g^{\gtrless}(\bar{\mathbf{r}}_1 t_1, 1')$$
$$+ \int_{t_0}^{t_1}\mathrm{d}\bar{1}[\Sigma^>(1,\bar{1}) - \Sigma^<(1,\bar{1})]g^{\gtrless}(\bar{1},1')$$
$$- \int_{t_0}^{t_1'}\mathrm{d}\bar{1}\Sigma^{\gtrless}(1,\bar{1})[g^>(\bar{1},1') - g^<(\bar{1},1')], \tag{6}$$

and have to be fulfilled together with the adjoint equation. The l.h.s. of Eq. (6) contains all single-particle terms, whereas the r.h.s. contains all corrections due to mean field, exchange and correlations. Further, t_0 denotes the initial time where the system is assumed to be uncorrelated (otherwise, the equations have to be supplemented with an initial correlation contribution to Σ, cf. [26,27]). Σ^{HF} is the Hartree–Fock self-energy (Hartree mean-field plus exchange energy) and $\Sigma^{\gtrless}$ are the correlation self-energies which describe collision processes, ionization and so on.

The equations of motion for the correlation functions (6) are completely general. They include an arbitrary electromagnetic field and confinement potential as well as any kind of correlation and scattering effects. These equations can be solved directly or used, via the reconstruction ansatz, to derive a closed equation for the density matrix (3). For the purposes here, where analogy with corresponding classical forms are exploited, an equivalent representation, the Wigner representation, is more appropriate. This is obtained by first introducing the center of mass and relative variables

$$\mathbf{R} = \frac{\mathbf{r}_1 + \mathbf{r}_1'}{2}, \quad T = \frac{t_1 + t_1'}{2}, \quad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_1', \quad \tau = t_1 - t_1'. \tag{7}$$

The Wigner representation for a function of $\mathbf{r}_1, \mathbf{r}'_1, t_1, t'_1$ is obtained by Fourier transform with respect to $\mathbf{r}$ in the new coordinates. For example, the density matrix $F_1\left(\mathbf{R}+\frac{\mathbf{r}}{2}, \mathbf{R}-\frac{\mathbf{r}}{2}, t\right) \equiv F(\mathbf{R}, \mathbf{r}, t)$, for which $\tau = 0$, $T = t$, becomes in the Wigner representation

$$f(\mathbf{p}, \mathbf{R}, t) = \pm \mathrm{i}\hbar g^<(\mathbf{p}, \mathbf{R}; t_1, t'_1)|_{t_1=t'_1=t}. \tag{8}$$

where $f(\mathbf{R}, \mathbf{p}, t)$ is the Fourier transform of $F(\mathbf{R}, \mathbf{r}, t)$. In the presence of external electromagnetic fields, it is useful to modify this Fourier transform to assure gauge invariance, as described in the next subsection.

2.2. Gauge-invariant Fourier transform of the KBE

It is well known that the electromagnetic field can be introduced in various ways (gauges) which may lead to essentially different explicit forms of the resulting kinetic equations. Although alternative derivations are successfully applied too, gauge invariance becomes a particular problem if the resulting kinetic equations are treated by means of approximations, such as retardation or gradient expansions, e.g. [16]. To keep the theory as general as possible and to avoid these difficulties, we will formulate the theory in terms of correlation functions which are made explicitly gauge-invariant.

In this section, we use a co-variant 4-vector notation as it makes the following transformations more compact and symmetric. The corresponding definitions are

$$A_\mu = (c\phi, \mathbf{A}), \quad x_\mu = (c\tau, \mathbf{r}), \quad X_\mu = (ct, \mathbf{R}),$$

and the conventions $a_\mu = (a_0, \mathbf{a}), a^\mu = (a_0, -\mathbf{a})$ and $a_\mu b^\mu = a_0 b_0 - \mathbf{ab}$ are being used.

The Kadanoff–Baym/Keldysh equations (6) remain covariant under gauge transformations $\chi(x)$, i.e. under the following transformations of the potentials and field operators [23,19]

$$\widetilde{A}_\mu(x) = A_\mu(x) - \partial_\mu \chi(x); \qquad \tilde{\psi}_a(x) = \mathrm{e}^{\frac{\mathrm{i}e_a}{\hbar c}\chi(x)}\psi_a(x), \tag{9}$$

leading to the gauge transform of the Green's functions

$$\tilde{g}_a(x, X) = \mathrm{e}^{\frac{\mathrm{i}e_a}{\hbar c}\left[\chi\left(X+\frac{x}{2}\right)-\chi\left(X-\frac{x}{2}\right)\right]} g_a(x, X).$$

Following an idea of Fujita [17], we introduce a gauge-invariant Green's function $\tilde{g}_a(\mathbf{k}, X)$ which is given by the modified Fourier transform

$$\tilde{g}(\mathbf{k}, X) = \int \frac{\mathrm{d}^4 x}{(2\pi)^4} \exp\left\{\mathrm{i}\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{d}\lambda x_\mu \left[k^\mu + \frac{e_a}{c}A^\mu(X+\lambda x)\right]\right\} g(x, X). \tag{10}$$

Then under any gauge transform (9) it is seen that the phase factors in the transformation to $\tilde{g}_a(x, X)$ cancel those in the transformation to $\widetilde{A}_\mu(x)$ in the modified Fourier transform, leading to the invariance $\tilde{g}_a(\mathbf{k}, X) \equiv g_a(\mathbf{k}, X)$.

In the following, we focus on spatially homogeneous electric fields and use the vector potential gauge

$$A_0 = \phi = 0; \qquad \mathbf{A} = -c\int_{-\infty}^{t} \mathrm{d}\bar{t}\ \mathbf{E}(\bar{t}). \tag{11}$$

In this case, the time part of the transform (10) becomes the usual Fourier transform, without any field modification. Applying first the spatial part of the transform (10) to Eq. (6) yields the gauge-invariant quantum kinetic equation in Wigner representation [23]

$$\begin{aligned}
&\left\{\mathrm{i}\hbar\frac{\partial}{\partial t_1} - \mathrm{i}\hbar\frac{\mathbf{K}_a^{\mathrm{A}}(t_1, t'_1)}{t_1 - t'_1}\nabla_{\mathbf{k}} - \epsilon\left[\frac{\hbar}{2\mathrm{i}}\nabla_{\mathbf{R}} + \mathbf{k} - \mathbf{K}_a^{\mathrm{A}}(t_1, t'_1)\right]\right\}\tilde{g}_a^{\gtrless}(\mathbf{k}, \mathbf{R}, t_1, t'_1) \\
&\quad - \int \mathrm{d}\mathbf{r}\int \frac{\mathrm{d}\mathbf{k}_1}{(2\pi\hbar)^3}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\mathbf{r}(\mathbf{k}-\mathbf{k}_1)} V^{\mathrm{eff}}\left(\mathbf{R}+\frac{\mathbf{r}}{2}, t_1\right)\tilde{g}_a^{\gtrless}(\mathbf{k}_1, \mathbf{R}, t_1, t'_1) \\
&\quad = \widetilde{I}_{\mathrm{F}_a}^{\gtrless}(\mathbf{k}, \mathbf{R}, t_1, t'_1) + \widetilde{I}_a^{\gtrless}(\mathbf{k}, \mathbf{R}, t_1, t'_1),
\end{aligned} \tag{12}$$

where the Hartree mean field Σ_a^{H} is local (momentum independent) and, therefore, can be included into the effective confinement potential

$$V^{\mathrm{eff}}(\mathbf{R},t) \equiv V(\mathbf{R},t) + \Sigma_a^{\mathrm{H}}(\mathbf{R},t). \tag{13}$$

The electromagnetic field enters Eq. (12) via the field-dependent momentum

$$\mathbf{K}_a^A(t_1,t_1') \equiv \frac{e_a}{c(t_1-t_1')}\int_{t_1'}^{t_1} \mathrm{d}t''[\mathbf{A}(t_1)-\mathbf{A}(t'')] = \frac{e_a}{t_1-t_1'}\int_{t_1'}^{t_1} \mathrm{d}t''(t_1'-t'')\mathbf{E}(t''). \tag{14}$$

Further, the r.h.s. of Eq. (12) contains all two-particle and higher order contributions to the dynamics in excess of the Hartree mean-field: the exchange contribution $\widetilde{I}_{Fa}^{\gtrless}$ and the collision integrals $\widetilde{I}_a^{\gtrless}$. The gauge-invariant collision integrals are

$$\begin{aligned}\widetilde{I}_a^{\gtrless} = &\int_{t_0}^{t_1} \mathrm{d}\bar{t}_1\Big\{\widetilde{\Sigma}_a^{>}(\mathbf{k}_1,\mathbf{R},t_1,\bar{t}_1) - \widetilde{\Sigma}_a^{<}(\mathbf{k}_1,\mathbf{R},t_1,\bar{t}_1)\Big\}\tilde{g}_a^{\gtrless}(\mathbf{k}_2,\mathbf{R},\bar{t}_1,t_1')\\ &- \int_{t_0}^{t_1'} \mathrm{d}\bar{t}_1\,\widetilde{\Sigma}_a^{\gtrless}(\mathbf{k}_1,\mathbf{R},t_1,\bar{t}_1)\{\tilde{g}_a^{>}(\mathbf{k}_2,\mathbf{R},\bar{t}_1,t_1') - \tilde{g}_a^{<}(\mathbf{k}_2,\mathbf{R},\bar{t}_1,t_1')\},\end{aligned} \tag{15}$$

with $\mathbf{k}_1 = \mathbf{k} + \frac{e_a}{c}\int_{t_1'}^{t_1}\mathrm{d}t'\frac{\mathbf{A}(t')}{t_1-t_1'} - \frac{e_a}{c}\int_{t_1}^{t_1}\mathrm{d}t'\frac{\mathbf{A}(t')}{t_1-t_1}$ and $\mathbf{k}_2 = \mathbf{k} + \frac{e_a}{c}\int_{t_1'}^{t_1}\mathrm{d}t'\frac{\mathbf{A}(t')}{t_1-t_1'} - \frac{e_a}{c}\int_{t_1'}^{\bar{t}_1}\mathrm{d}t'\frac{\mathbf{A}(t')}{\bar{t}_1-t_1'}$. The equations of motion (12) and the adjoint for the two-time correlation functions are the basis for a very general analysis of the nonequilibrium behavior of confined quantum particles in a strong time-dependent electric field. In principle, their direct numerical solution is possible, following previous numerical works for spatially homogeneous charged particle systems, e.g. [14,28,29]. Alternatively, one can use these equations as a starting point for analytical derivations. In particular, these equations directly yield the gauge-invariant equation for the Wigner distribution function which was obtained in Refs. [23,24].

2.3. Spectral properties of particles in a strong field

The strength of the Green's functions approach is that both statistical and dynamical properties, described by the Wigner function and the spectral function, are treated self-consistently. This is of particular importance if approximations are being developed. In this section, the equation for the spectral function is obtained in the Wigner representation. The resulting approximate spectral function obtained is then used to evaluate the collision operator of the kinetic equation in Section 4.

The spectral information is obtained by considering the Green's functions $\tilde{g}_a^{\gtrless}(\mathbf{k},\mathbf{R},t_1,t_1')$ as a function of the relative time τ, where $t=(t_1+t_1')/2$, $\tau = t_1 - t_1'$. This is determined by considering the sum of Eq. (12) and its adjoint, and using the definition for the spectral function (4) to get

$$\begin{aligned}&\Big\{\mathrm{i}\hbar\frac{\partial}{\partial\tau} - \frac{\mathrm{i}\hbar}{\tau}\mathbf{K}_{a+}^{\mathrm{A}}(t_1,t_1')\nabla_{\mathbf{k}} - \frac{1}{2m_a}\left[\left(\frac{\hbar}{2i}\nabla_{\mathbf{R}}\right)^2 + k^2 - \frac{\hbar}{i}\mathbf{K}_{a-}^{\mathrm{A}}(t_1,t_1')\nabla_{\mathbf{R}}\right.\\ &\quad\left.+\frac{1}{2}\left(\mathbf{K}_a^{\mathrm{A}2}(t_1,t_1') + \mathbf{K}_a^{\mathrm{A}2}(t_1',t_1)\right) - 2\mathbf{k}\mathbf{K}_{a+}^{\mathrm{A}}(t_1,t_1')\right]\Big\}a(\mathbf{k},\mathbf{R},\tau,t)\\ &\quad-\frac{1}{2}\int\mathrm{d}\mathbf{r}\int\frac{\mathrm{d}\mathbf{k}_1}{(2\pi\hbar)^3}\Big\{\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\mathbf{r}(\mathbf{k}-\mathbf{k}_1)}V^{\mathrm{eff}}\Big(\mathbf{R}+\frac{\mathbf{r}}{2},t_1\Big) + \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathbf{r}(\mathbf{k}-\mathbf{k}_1)}V^{\mathrm{eff}}\Big(\mathbf{R}+\frac{\mathbf{r}}{2},t_1'\Big)\Big\}a(\mathbf{k}_1,\mathbf{R},\tau,t)\\ &\quad=-\hbar\mathrm{Im}\Big(\widetilde{I}_{\mathrm{F}a}^{>}(\mathbf{k},\mathbf{R},t_1,t_1') - \widetilde{I}_{\mathrm{F}a}^{<}(\mathbf{k},\mathbf{R},t_1,t_1') + \widetilde{I}_a^{>}(\mathbf{k},\mathbf{R},t_1,t_1') - \widetilde{I}_a^{<}(\mathbf{k},\mathbf{R},t_1,t_1')\Big),\end{aligned} \tag{16}$$

where a parabolic dispersion relation has been used. The vectors $\mathbf{K}_{a\pm}^A(t_1,t_1')$ are related to that of (14) by

$$\mathbf{K}_{a\pm}^A(t_1,t_1') \equiv \frac{1}{2}\left[\mathbf{K}_a^A(t_1,t_1') \pm \mathbf{K}_a^A(t_1',t_1)\right] = \frac{e_a}{2\tau}\int_{t-\frac{\tau}{2}}^{t+\frac{\tau}{2}}\mathrm{d}t''\left[\left(t-\frac{\tau}{2}-t''\right) \pm \left(t+\frac{\tau}{2}-t''\right)\right]\mathbf{E}(t''). \tag{17}$$

The result (16) is still completely general, including strong field effects, spatial inhomogeneities and correlations.

The left side of (16) describes the "kinetics" of the spectral function with respect to τ. In addition to the effects of the external electromagnetic field this describes the single particle effects of the kinetic energy and the effective potential $V^{\text{eff}}(\mathbf{R}+\frac{\mathbf{r}}{2},t_1)$. This is non-local due to quantum effects resulting from the non-commutation of kinetic and potential energy. However, if this confining potential is constant, or sufficiently smooth, in both space and time then a local approximation can be made

$$\frac{1}{2}\int \mathrm{d}\mathbf{r}\int\frac{\mathrm{d}\mathbf{k}_1}{(2\pi\hbar)^3}\left\{\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\mathbf{r}(\mathbf{k}-\mathbf{k}_1)}V^{\text{eff}}\left(\mathbf{R}+\frac{\mathbf{r}}{2},t_1\right)+\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathbf{r}(\mathbf{k}-\mathbf{k}_1)}V^{\text{eff}}\left(\mathbf{R}+\frac{\mathbf{r}}{2},t_1'\right)\right\}a(\mathbf{k}_1,\mathbf{R},\tau,t)$$
$$\to V(\mathbf{R},t)a(\mathbf{k},\mathbf{R},\tau,t). \tag{18}$$

Also in this limit the nonlocal terms proportional to gradients of $a(\mathbf{k},\mathbf{R},\tau,t)$ can be neglected. The resulting local form can be solved exactly in the collisionless limit to get

$$a(\mathbf{k},\mathbf{R},\tau,t;E)=\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\left(\left(\frac{k^2}{2m_a}+V^{\text{eff}}\left(\mathbf{R}+\frac{\mathbf{r}}{2},t_1\right)\right)\tau+\frac{e_a^2}{2m_a}\left[\int_{t-\frac{\tau}{2}}^{t+\frac{\tau}{2}}\mathrm{d}\bar{t}\left(\int_t^{\bar{t}}\mathrm{d}t''\mathbf{E}(t'')\right)^2-\frac{1}{\tau}\left(\int_{t-\frac{\tau}{2}}^{t+\frac{\tau}{2}}\mathrm{d}\bar{t}\int_t^{\bar{t}}\mathrm{d}t''\mathbf{E}(t'')\right)^2\right]\right)}. \tag{19}$$

In the absence of external fields, the corresponding spectral density (Fourier transform with respect to τ) is the expected sharp result

$$a(\mathbf{k},\mathbf{R},\omega,t;\mathbf{A})\to 2\pi\hbar\delta(\hbar\omega-\epsilon(\mathbf{k},\mathbf{R})),\quad \epsilon(\mathbf{k},\mathbf{R})=\frac{k^2}{2m_a}+V^{\text{eff}}(\mathbf{R},t). \tag{20}$$

More generally, much of the simplicity of the local form can be retained without losing the important quantum effects by the introduction of a "quantum potential" $V_{\text{Q}}^{\text{eff}}(\mathbf{k},\mathbf{R},\tau,t)$ that absorbs all non-local terms of (16). This is accomplished by writing (16) in the form

$$\left\{\mathrm{i}\hbar\left(\frac{\partial}{\partial\tau}-\frac{1}{\tau}\mathbf{K}_{a+}^{\text{A}}(t_1,t_1')\nabla_{\mathbf{k}}\right)-\left[\frac{k^2}{2m_a}+V_{\text{Q}}^{\text{eff}}(\mathbf{k},\mathbf{R},\tau,t)+\frac{1}{2}\left(\mathbf{K}_a^{\text{A}^2}(t_1,t_1')+\mathbf{K}_a^{\text{A}^2}(t_1',t_1)\right)-2\mathbf{k}\mathbf{K}_{a+}^{\text{A}}(t_1,t_1')\right]\right\}a$$
$$=-\hbar\mathrm{Im}\left(\widetilde{I}_{\text{F}_a}^{>}(\mathbf{k},\mathbf{R},t_1,t_1')-\widetilde{I}_{\text{F}_a}^{<}(\mathbf{k},\mathbf{R},t_1,t_1')+\widetilde{I}_a^{>}(\mathbf{k},\mathbf{R},t_1,t_1')-\widetilde{I}_a^{<}(\mathbf{k},\mathbf{R},t_1,t_1')\right), \tag{21}$$

where $a=a(\mathbf{k},\mathbf{R},\tau,t)$. This will be exactly equivalent to (16) if the quantum potential is defined by

$$V_{\text{Q}}^{\text{eff}}(\mathbf{k},\mathbf{R},\tau,t)\equiv\frac{1}{2}\int\mathrm{d}\mathbf{r}\int\frac{\mathrm{d}\mathbf{k}_1}{(2\pi\hbar)^3}\left\{\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\mathbf{r}(\mathbf{k}-\mathbf{k}_1)}V^{\text{eff}}\left(\mathbf{R}+\frac{\mathbf{r}}{2},t_1\right)+\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathbf{r}(\mathbf{k}-\mathbf{k}_1)}V^{\text{eff}}\left(\mathbf{R}+\frac{\mathbf{r}}{2},t_1'\right)\right\}\frac{a(\mathbf{k}_1,\mathbf{R},\tau,t)}{a(\mathbf{k},\mathbf{R},\tau,t)}$$
$$+\frac{1}{2m_a}\left[\left(\frac{\hbar}{2i}\right)^2\left((\nabla_R\ln a(\mathbf{k},\mathbf{R},\tau,t))^2+\nabla_R^2\ln a(\mathbf{k},\mathbf{R},\tau,t)\right)\right.$$
$$\left.-\frac{\hbar}{i}\mathbf{K}_{a-}^{A}(t_1,t_1')\cdot\nabla_R\ln a(\mathbf{k},\mathbf{R},\tau,t)\right] \tag{22}$$

However, its local form is now more suitable for approximations. For example, in the collisionless limit the solution to (21) is found to be

$$a(\mathbf{k},\mathbf{R},\tau,t;E)=\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\left(\frac{k^2}{2m_a}\tau+\int_0^\tau\mathrm{d}\tau' V_{\text{Q}}^{\text{eff}}(\mathbf{k}+\kappa(\tau,\tau',t),\mathbf{R},\tau',t)+\frac{e_a^2}{2m_a}\left[\int_{t-\frac{\tau}{2}}^{t+\frac{\tau}{2}}\mathrm{d}\bar{t}\left(\int_t^{\bar{t}}\mathrm{d}t''\mathbf{E}(t'')\right)^2-\frac{1}{\tau}\left(\int_{t-\frac{\tau}{2}}^{t+\frac{\tau}{2}}\mathrm{d}\bar{t}\int_t^{\bar{t}}\mathrm{d}t''\mathbf{E}(t'')\right)^2\right]\right)}. \tag{23}$$

where the momentum dependence of the quantum potential appears shifted by $\boldsymbol{\kappa}(\tau,t)$

$$\boldsymbol{\kappa}(\tau,\tau',t)\equiv\int_{\tau'}^{\tau}\mathrm{d}\tau''\frac{1}{\tau''}\mathbf{K}_{a+}^{A}\left(t+\frac{\tau''}{2},t-\frac{\tau''}{2}\right)=\frac{e_a}{2}\int_{\tau'}^{\tau}\mathrm{d}\tau''\frac{1}{\tau''^2}\int_{t-\frac{\tau''}{2}}^{t+\frac{\tau''}{2}}\mathrm{d}t'''(t-t''')\mathbf{E}(t'''). \tag{24}$$

For our present analysis, where we are interested in the effect of intense electric fields and quantum effects of the confinement potential on the spectrum, these are all accounted for in this simple form. As was found in many investigations, in the case of strong fields inclusion of all external fields into the spectral function together with Hartree–Fock effects is crucial for a correct modeling of the many-par-

ticle behavior; see e.g. [29]. We expect that a detailed self-consistent treatment of the collision and correlation effects on the spectral function can be avoided. Correlation effects lead to an energy shift and broadening which is generally well understood. These effects can be added to the collisionless result in perturbation theory. Therefore, although the introduction of the quantum potential in (21) is general, the collisionless solution (23) is a useful practical result as is illustrated in the following sections.

3. Approximate quantum potential

It remains to determine an explicit form for the quantum potential from an approximation to the definition (22). The definition is only implicit since it depends on the unknown spectral density. A self-consistent equation in the collisionless limit is obtained by using the result (23) in (22)

$$\begin{aligned} V_{\mathrm{Q}}^{\mathrm{eff}}(\mathbf{k},\mathbf{R},\tau,t) \equiv \frac{1}{2}\int \mathrm{d}\mathbf{r}\int \frac{\mathrm{d}\mathbf{k}_1}{(2\pi\hbar)^3}\left\{\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\mathbf{r}(\mathbf{k}-\mathbf{k}_1)}V^{\mathrm{eff}}\left(\mathbf{R}+\frac{\mathbf{r}}{2},t_1\right)+\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathbf{r}(\mathbf{k}-\mathbf{k}_1)}V^{\mathrm{eff}}\left(\mathbf{R}+\frac{\mathbf{r}}{2},t_1'\right)\right\} \\ \times\exp\left\{-\frac{\mathrm{i}}{\hbar}\left(\frac{\tau}{2m_a}(k_1^2-k^2)+\int_0^\tau \mathrm{d}\tau'\left(V_{\mathrm{Q}}^{\mathrm{eff}}(\mathbf{k}_1+\kappa(\tau,\tau',t),\mathbf{R},\tau',t)\right.\right.\right. \\ \left.\left.\left.-V_{\mathrm{Q}}^{\mathrm{eff}}(\mathbf{k}+\kappa(\tau,\tau',t),\mathbf{R},\tau',t)\right)\right)\right\}+\frac{1}{2m_a}\left[\frac{1}{4}\left(\int_0^\tau \mathrm{d}\tau'\nabla_R V_{\mathrm{Q}}^{\mathrm{eff}}(\mathbf{k}+\kappa(\tau,\tau',t),\mathbf{R},\tau',t)\right)^2\right. \\ +\frac{\mathrm{i}\hbar}{4}\int_0^\tau \mathrm{d}\tau'\nabla_R^2 V_{\mathrm{Q}}^{\mathrm{eff}}(\mathbf{k}+\kappa(\tau,\tau',t),\mathbf{R},\tau',t)-\mathbf{K}_{a-}^A(t_1,t_1') \\ \left.\cdot\int_0^\tau \mathrm{d}\tau'\nabla_R V_{\mathrm{Q}}^{\mathrm{eff}}(\mathbf{k}+\kappa(\tau,\tau',t),\mathbf{R},\tau',t)\right] \end{aligned} \tag{25}$$

Here, we obtain a practical solution to this equation by considering first the limit $\hbar\to 0$. This is similar in spirit to the time independent WKB approximation. The details are described in Appendix A with the final result

$$\begin{aligned} V_{\mathrm{Q}}^{\mathrm{eff}}(\mathbf{k},\mathbf{R},\tau,t;\hbar=0) = \frac{1}{2}\left(V^{\mathrm{eff}}\left(\mathbf{R}+\mathbf{z}(\mathbf{k},\mathbf{R},t),t+\frac{\tau}{2}\right)+V^{\mathrm{eff}}\left(\mathbf{R}-\mathbf{z}(\mathbf{k},\mathbf{R},t),t-\frac{\tau}{2}\right)\right) \\ +\frac{1}{8m_a}\left(\int_0^\tau \mathrm{d}\tau'\nabla_R V_{\mathrm{Q}}^{\mathrm{eff}}(\mathbf{k}+\kappa(\tau,\tau',t),\mathbf{R},\tau',t;\hbar=0)\right)^2 \\ +\frac{1}{2m_a}K_{a-}^A(t_1,t_1')\cdot\int_0^\tau \mathrm{d}\tau'\nabla_R V_{\mathrm{Q}}^{\mathrm{eff}}(\mathbf{k}+\kappa(\tau,\tau',t),\mathbf{R},\tau',t;\hbar=0) \end{aligned} \tag{26}$$

with

$$\mathbf{z}(\mathbf{k},\mathbf{R},\tau,t)=\nabla_k\phi(\mathbf{k},\mathbf{R},\tau,t)=\tau\frac{\mathbf{k}}{m_a}+\int_0^\tau \mathrm{d}\tau'\nabla_k V_{\mathrm{Q}}^{\mathrm{eff}}(\mathbf{k}+\kappa(\tau,\tau',t),\mathbf{R},\tau',t;\hbar=0). \tag{27}$$

Next, we assume $V^{\mathrm{eff}}\left(\mathbf{R}+\frac{\mathbf{r}}{2},t_1\right)$ is a smooth function of $\mathbf{r}$ so that an expansion about $\mathbf{R}$ can be performed. This in turn induces an expansion of $V_{\mathrm{Q}}^{\mathrm{eff}}$ in the gradients of V^{eff}

$$V_{\mathrm{Q}}^{\mathrm{eff}}(\mathbf{k},\mathbf{R},\tau,t)=V_{\mathrm{Q}}^{(0)}(\mathbf{k},\mathbf{R},\tau,t)+\lambda V_{\mathrm{Q}}^{(1)}(\mathbf{k},\mathbf{R},\tau,t)+\lambda^2 V_{\mathrm{Q}}^{(2)}(\mathbf{k},\mathbf{R},\tau,t)+\cdots, \tag{28}$$

where the formal parameter λ (set equal to unity at the end) is introduced to represent the order of the gradient in V^{eff}. The coefficients $V_{\mathrm{Q}}^{(0)}$ are identified by substitution of (28) into (27), expanding $V^{\mathrm{eff}}\left(\mathbf{R}\pm\mathbf{z}(\mathbf{k},\mathbf{R},t),t\pm\frac{\tau}{2}\right)$ in a similar way, and equating powers of λ. The analysis is straightforward for the general case, but to simplify the results, in the following, we will assume a slowly varying in time confining potential for which $\tau\partial V^{\mathrm{eff}}(\mathbf{R},t)/\partial t \ll V^{\mathrm{eff}}(\mathbf{R},t)$ and consequently set $V^{\mathrm{eff}}\left(\mathbf{R}\pm\mathbf{z}(\mathbf{k},\mathbf{R},t),t\pm\frac{\tau}{2}\right)\to V^{\mathrm{eff}}(\mathbf{R}\pm\mathbf{z}(\mathbf{k},\mathbf{R},t),t)$. Furthermore, we consider the case of a constant electric field $\mathbf{E}(t)=\mathbf{E}$. In this case, the quantum potential up through second order is found to be (see Appendix A for details)

$$V_{\mathrm{Q}}^{\mathrm{eff}}(\mathbf{k},\mathbf{R},\tau,t;\hbar=0)\to V^{\mathrm{eff}}(\mathbf{R},t)+\frac{\tau^2}{8m_a^2}\left[\left(\nabla_R V^{\mathrm{eff}}(\mathbf{R},t)\right)^2+(\mathbf{k}\cdot\nabla_R)^2V^{\mathrm{eff}}(\mathbf{R},t)-2e_a\mathbf{E}\cdot\nabla_R V^{\mathrm{eff}}(\mathbf{R},t)\right]$$
$$+\frac{\tau^4}{3}\left(\frac{e_a}{4m_a}\mathbf{E}\cdot\nabla_R\right)^2V^{\mathrm{eff}}(\mathbf{R},t). \tag{29}$$

These results provide the explicit form for the quantum potential, exact in the collisionless limit at $\hbar=0$ up through second order in the gradients of the confining potential. It describes both the leading non-local effects and their coupling to arbitrary external fields.

Use of this approximate quantum potential in (23) gives the corresponding self-consistent spectral function. Calculation of the spectral function in (23) requires evaluation of $\int_0^\tau \mathrm{d}\tau' V_{\mathrm{Q}}^{\mathrm{eff}}(\mathbf{k}+\kappa(\tau,t),\mathbf{R},\tau',t)$. It follows from the definition in (24) that $\boldsymbol{\kappa}(\tau,\tau',t)$ vanishes for a constant electric field. The spectral function is then found to be

$$a(\mathbf{k},\mathbf{R},\tau,t;)\to \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\left(\left(\frac{k^2}{2m_a}+V^{\mathrm{eff}}(\mathbf{R},t)\right)\tau+X(\mathbf{k},\mathbf{R},t;\mathbf{E})\tau^3+Y(\mathbf{R},t;\mathbf{E})\tau^5\right)}, \tag{30}$$

with

$$X(\mathbf{k},\mathbf{R},t;\mathbf{E})=\frac{1}{24m_a}\left(\left(\nabla_R V^{\mathrm{eff}}(\mathbf{R},t)\right)^2+\frac{1}{m_a}(\mathbf{k}\cdot\nabla_R)^2V^{\mathrm{eff}}(\mathbf{R},t)-2e_a\mathbf{E}\cdot\nabla_R V^{\mathrm{eff}}(\mathbf{R},t)+(e_aE)^2\right) \tag{31}$$

$$Y(\mathbf{R},t;\mathbf{E})=\frac{1}{240m_a^2}(e_a\mathbf{E}\cdot\nabla_R)^2V^{\mathrm{eff}}(\mathbf{R},t). \tag{32}$$

The corresponding spectral density is obtained by Fourier transformation with respect to τ

$$a(\mathbf{k},\mathbf{R},\omega,t;\mathbf{E})=\frac{2\pi\hbar}{\alpha}I\left\{\frac{\left[\left(\frac{k^2}{2m_a}+V^{\mathrm{eff}}(\mathbf{R},t)\right)-\hbar\omega\right]}{\alpha},\beta\right\}, \tag{33}$$

where

$$I(x,y)=\frac{1}{\pi}\int_0^\infty \mathrm{d}\tau\cos\left(x\tau+\frac{1}{3}\tau^3+(y\tau)^5\right), \tag{34}$$

and

$$\alpha=(3\hbar^2X(\mathbf{k},\mathbf{R},t;\mathbf{E}))^{1/3},\quad \beta=\frac{1}{\alpha}(\hbar^4Y(\mathbf{R},t;\mathbf{E}))^{1/5}. \tag{35}$$

This is a simple result easily explored numerically. Some analytic limits follow directly. In the absence of both gradients of $V^{\mathrm{eff}}(\mathbf{R},t)$ and electromagnetic fields, $\alpha,\beta\to 0$, and the sharp delta function result of Eq. (20) is recovered. If gradients of $V^{\mathrm{eff}}(\mathbf{R},t)$ are neglected but an arbitrary constant field is applied, $\alpha\to((\hbar e_aE)^2/8m_a)^{1/3}$, $\beta\to 0$, and

$$a(\mathbf{k},\mathbf{R},\omega,t;\mathbf{E})\to\frac{2\pi\hbar}{a}\mathrm{Ai}\left\{\frac{\left[\frac{k^2}{2m_a}+V^{\mathrm{eff}}-\hbar\omega\right]}{\alpha}\right\}, \tag{36}$$

where $\mathrm{Ai}(x)$ is the Airy function. It is seen that the electric field broadens the sharp distribution, but it is still peaked at $\hbar\omega=\frac{k^2}{2m_a}+V^{\mathrm{eff}}$. Finally, in the absence of electric fields, $\beta\to 0$, the quantum effects represented by the spatial variation of $V^{\mathrm{eff}}(\mathbf{R},t)$ leads to the same result as (36), but with

$$\alpha\to\left[\frac{\hbar^2}{8m_a}\left(\frac{1}{m_a}(\mathbf{k}\cdot\nabla_R)^2V^{\mathrm{eff}}(\mathbf{R},t)+\left(\nabla_R V^{\mathrm{eff}}(\mathbf{R},t)\right)^2\right)\right]^{1/3}. \tag{37}$$

This is shown in Fig. 1.

Interestingly, the quantum correction to the potential enters into the spectral function in the same way as the constant electric field. Both lead to a broadening of the main peak of the spectral function around $\hbar\omega$, i.e. the sharp single-particle energy is smeared out, so that the Heisenberg uncertainty principle is no longer violated. In addition there arise smaller side peaks. The bigger α, the more the spectral function shifts towards higher energies. This is readily understood. The electric field accelerates the electrons and thus increases their energy. Similarly in an inhomogeneous confinement potential a quantum particle acquires an additional kinetic energy which arises from spatial compression of its wave function, which is proportional to the local curvature of V^{eff}. Additionally, for bigger α the peaks become lower and broader. Because of this also energies below the classical single-particle energy $\frac{k^2}{2m_a}+V^{\text{eff}}$ become possible. Note that there occur negative values of the spectral function. Although this is unexpected [16], this is not in contrast with sum rules. In particular the sum rule $\int \mathrm{d}\omega a(\omega)=2\pi$ is fulfilled for (33).

3.1. Density of states

With the knowledge of the spectral function one can directly calculate the density of states (DOS) using in the 3D case a parabolic dispersion:

$$\rho(\omega)=-\frac{1}{\pi}\mathrm{Im}\left[\mathrm{Tr}G^{\mathrm{R}}(\mathbf{k},\mathbf{R},\omega)\right]=\frac{1}{(2\pi\hbar)^4}\int \mathrm{d}\mathbf{k}\int \mathrm{d}\mathbf{R}a(\mathbf{k},\mathbf{R},\omega,t;\mathbf{E})$$
$$=\frac{\rho_0}{8\pi^2\hbar}\int \mathrm{d}\hat{\mathbf{k}}\int \mathrm{d}\mathbf{R}\int_0^\infty \mathrm{d}\epsilon\,\epsilon^{1/2}a(\epsilon,\mathbf{R},\omega,t;\mathbf{E}), \tag{38}$$

with $\rho_0=4\sqrt{2}\pi m_a^{3/2}/(2\pi\hbar)^3$ and the unit vector $\widehat{\mathbf{k}}\equiv\mathbf{k}/k$. Consider first the local approximation with the spectral density given by (20)

$$a(\mathbf{k},\mathbf{R},\omega,t;\mathbf{E})=2\pi\hbar\,\delta(\epsilon+V(\mathbf{R},t)-\hbar\omega),$$

to get

$$\rho(\omega)\to\rho^{(0)}(\omega)=4\pi\rho_0\Theta(\omega)\int_0^{R_m(\hbar\omega)}\mathrm{d}RR^2(V^{\text{eff}}(R_m)-V^{\text{eff}}(\mathbf{R}))^{1/2}, \tag{39}$$

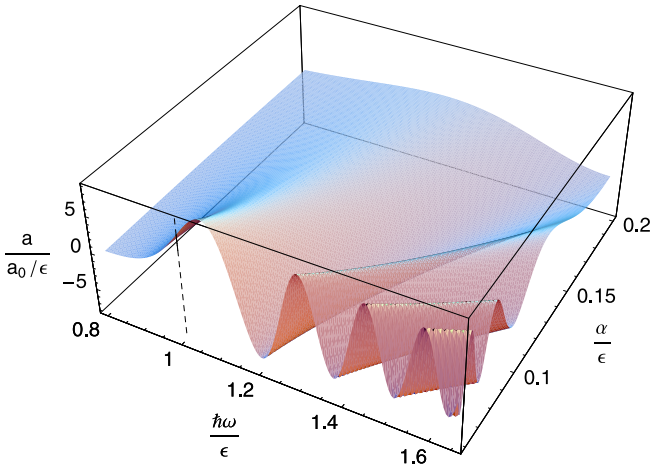

Fig. 1. Spectral function $a(\mathbf{k},\mathbf{R},\omega,t;\alpha)$, Eq. (36), at fixed time t and a fixed phase space point $(\mathbf{k},\mathbf{R})$ such that the dimensionless local single-particle energy $\epsilon_a(\mathbf{k},\mathbf{R},t)=1$. The ideal spectral function would have a singularity at $\hbar\omega=1$ (dashed line), which is observed in the limit of vanishing inhomogeneity $\alpha\equiv(3\hbar^2\delta V_{\mathrm{Q}})^{\frac{1}{3}}\to 0$. With increasing inhomogeneity, due to the Heisenberg uncertainty principle, the main peak broadens and shifts towards higher energies.

where $R_m(\hbar\omega)$ is determined from the solution to

$$V^{\text{eff}}(R_m) = \hbar\omega. \tag{40}$$

Next, consider the addition of a homogeneous electric field but without the quantum confinement effects due to gradients of the potential. In this case, the spectral function is given by (36)

$$a(\mathbf{k},\mathbf{R},\omega,t) = \frac{2\pi\hbar}{\left[\hbar^2\frac{e_a^2E_0^2}{8m_a}\right]^{\frac{1}{3}}}\text{Ai}\left\{\frac{[\epsilon + V^{\text{eff}} - \hbar\omega]}{\left[\hbar^2\frac{e_a^2E_0^2}{8m_a}\right]^{\frac{1}{3}}}\right\}, \tag{41}$$

leading to the DOS

$$\rho(\omega) \to \int_{-\infty}^{\infty} \mathrm{d}z\rho^{(0)}(z)\frac{\hbar}{\left[\hbar^2\frac{e_a^2E_0^2}{8m_a}\right]^{\frac{1}{3}}}\text{Ai}\left(\frac{\hbar(z-\omega)}{\left[\hbar^2\frac{e_a^2E_0^2}{8m_a}\right]^{\frac{1}{3}}}\right), \tag{42}$$

where $\rho^{(0)}(z)$ is the local form (39). To illustrate in more detail consider the special case of a harmonic oscillator potential $V^{\text{eff}}(\mathbf{R}) = \frac{m_a}{2}\Omega^2R^2$. In this case, (39) becomes

$$\rho^{(0)}(\omega) = \frac{1}{2\hbar\Omega}\theta^2\Theta(\theta), \quad \theta = \frac{\omega}{\Omega}. \tag{43}$$

A numerical evaluation of (42) then yields the result shown in Fig. 2 for varying intensities of the electric field strengths. One can see that through the presence of a constant electric field there exist non-vanishing contributions of the density of states for negative energies, the bigger, the higher the intensity of the field is. Looking at the Hamiltonian of such a system this is easily explained:

$$\widehat{H}(p,q) = \frac{p^2}{2m_a} + \frac{m_a}{2}\Omega^2q^2 + e_0Eq = \frac{p^2}{2m_a} + \frac{m_a}{2}\Omega^2\left(q + \frac{e_0E}{m_a\Omega^2}\right)^2 - \frac{e_0^2E^2}{2m_a\Omega^2}. \tag{44}$$

A constant electric field shifts the energy minimum from 0 to $-\frac{e_0^2E^2}{2m_a\Omega^2}$, so that negative energy values become possible [16].

Now let us analyze the change of the DOS due to quantum confinement effects. Therefore, we compare the classical density of states (39) with the DOS that includes the non-local correction (37). Using the spectral function (36) with α given by (37) and the harmonic oscillator confining potential we obtain for the field free case

$$\rho(\theta) = \frac{2}{\pi\hbar\Omega}\int_{-1}^{1}\mathrm{d}x\int_0^{\infty}\mathrm{d}a\int_0^{\infty}\mathrm{d}b\,\frac{(ab)^{1/2}}{\left[\frac{1}{4}(x^2a+b)\right]^{1/3}}\text{Ai}\left\{\frac{[a+b-\theta]}{\left[\frac{1}{4}(x^2a+b)\right]^{1/3}}\right\}, \tag{45}$$

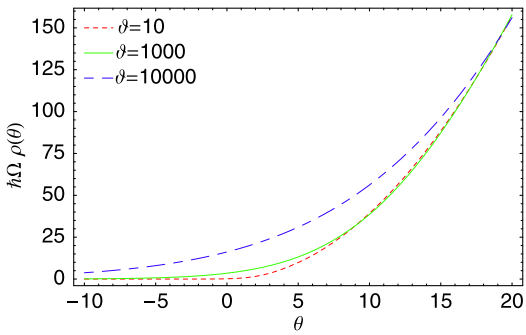

Fig. 2. Density of states for a classical particle system in a constant electric field and a harmonic oscillator potential for different dimensionless field strengths, where $\vartheta = \frac{e_0^2E_0^2}{m\Omega^2}, \frac{1}{\hbar\Omega}$.

where the change of variables $a = \frac{\epsilon}{\hbar\Omega}$ and $b = \frac{V(\mathbf{R})}{\hbar\Omega}$ has been made. This integral can be computed numerically. The consideration of a quantum potential leads – analogously to a constant electric field – to finite values of the DOS at small negative energies. More generally, since the modifications of the quantum potential have the same effect on the spectral function as the constant electric field and therefore causes the same behavior of the DOS. However, a simple interpretation of this effect in terms of a Hamiltonian like in (44) is not possible in this case.

4. Kinetic equation for the distribution function

In this section, we analyze the effect of the quantum potential of the last section on the collision integral in the kinetic equation for the Wigner distribution. The gauge-invariant kinetic equation for the single-time distribution function is derived by computing the difference of the gauge-invariant kinetic equation (12) and its adjoint for $g^<$ taken at equal times $t_1 = t_1'$. Relation (3) then allows us to express $g^<$ in terms of the Wigner distribution [23]

$$\left\{\frac{\partial}{\partial t} + \frac{\mathbf{k}}{m_a}\nabla_{\mathbf{R}} + e_a\mathbf{E}(t)\nabla_{\mathbf{k}}\right\}\tilde{f}_a(\mathbf{k},\mathbf{R},t) + \frac{2}{\hbar}\int \mathrm{d}\mathbf{r}\int \frac{\mathrm{d}\mathbf{k}_1}{(2\pi\hbar)^3}\sin\left(\frac{\mathbf{r}\cdot(\mathbf{k}-\mathbf{k}_1)}{\hbar}\right) \times V^{\mathrm{eff}}\left(\mathbf{R}+\frac{\mathbf{r}}{2},t\right)\tilde{f}_a(\mathbf{k}_1,\mathbf{R},t) = \widetilde{I}_a(\mathbf{k},\mathbf{R},t) + \widetilde{I}_{F_a}(\mathbf{k},\mathbf{R},t), \tag{46}$$

where $\widetilde{I}_a(\mathbf{k},\mathbf{R},t) \equiv \pm 2\mathrm{Re}[\widetilde{I}_a^<(\mathbf{k},\mathbf{R},t)]$ and $\widetilde{I}_{F_a}(\mathbf{k},\mathbf{R},t) \equiv \pm 2\mathrm{Re}[\widetilde{I}_{F_a}^<(\mathbf{k},\mathbf{R},t)]$. Here $\mathbf{E}$ is the total electric field (external plus induced) that obeys Maxwell's equations, which have to be solved self-consistently with the kinetic equation (46).

The non-local contribution from V^{eff} in Eq. (46) is still completely general and does not use any assumptions on the space dependence of the confinement potential. The present form applies even to sharp spatial changes and can be used with any form of pseudo-potentials. Analytical simplifications are possible if the potential is only weakly inhomogeneous. Then one can eliminate the integrations by expanding V^{eff} around the center of mass coordinate, e.g. [14]. With this, the integral term in Eq. (46) becomes

$$\frac{2}{\hbar}\int \mathrm{d}^3r\int \frac{\mathrm{d}^3k_1}{(2\pi\hbar)^3}\sin\left(\frac{\mathbf{r}\cdot(\mathbf{k}-\mathbf{k}_1)}{\hbar}\right)V^{\mathrm{eff}}\left(\mathbf{R}+\frac{\mathbf{r}}{2},t\right)\tilde{f}_a(\mathbf{k}_1,\mathbf{R},t) = -(\nabla_{\mathbf{R}}V^{\mathrm{eff}}(\mathbf{R},t))\cdot\nabla_{\mathbf{k}}\tilde{f}_a(\mathbf{k},\mathbf{R},t) + O(\hbar^2V'''^{\mathrm{eff}}). \tag{47}$$

As in the last section the non-local quantum effects can be incorporated in the classical form by writing (46) in the equivalent form

$$\left\{\frac{\partial}{\partial t} + \frac{\mathbf{k}}{m_a}\nabla_{\mathbf{R}} + e_a\mathbf{E}(t)\nabla_{\mathbf{k}} - (\nabla_{\mathbf{R}}\mathscr{V}_{\mathrm{Q}}^{\mathrm{eff}}(\mathbf{R},t))\cdot\nabla_{\mathbf{k}}\right\}\tilde{f}_a(\mathbf{k},\mathbf{R},t) = \widetilde{I}_a(\mathbf{k},\mathbf{R},t) + \widetilde{I}_{F_a}(\mathbf{k},\mathbf{R},t),$$

with the identification of $\mathscr{V}_{\mathrm{Q}}^{\mathrm{eff}}(\mathbf{R},t)$ by

$$(\nabla_{\mathbf{R}}\mathscr{V}_{\mathrm{Q}}^{\mathrm{eff}}(\mathbf{R},\mathbf{k},t))\cdot\nabla_{\mathbf{k}}\tilde{f}_a(\mathbf{k},\mathbf{R},t) = \frac{2}{\hbar}\int \mathrm{d}^3r\int \frac{\mathrm{d}^3k_1}{(2\pi\hbar)^3}\sin\left(\frac{\mathbf{r}\cdot(\mathbf{k}-\mathbf{k}_1)}{\hbar}\right)V^{\mathrm{eff}}\left(\mathbf{R}+\frac{\mathbf{r}}{2},t\right)\tilde{f}_a(\mathbf{k}_1,\mathbf{R},t). \tag{48}$$

Clearly, the requirements of a local classical form in the equation for the spectral function and that for the kinetic equation lead to different quantum potentials in each case: Eqs. (22) and (48). The quantum potential (48) for the kinetic equation is more formal since it depends on the specific nonequilibrium state considered. Still it is a potentially useful concept if there is a relevant reference state. For example, if the states considered are near equilibrium $\tilde{f}_a(\mathbf{k}_1,\mathbf{R},t)$ can be represented by the equilibrium distribution. More practically, this can be chosen as the ideal gas equilibrium state (first order perturbation theory in the potential) to give

$$(\nabla_{\mathbf{R}}\mathscr{V}_{\mathrm{Q}}^{\mathrm{eff}}(\mathbf{R},\mathbf{k},t))\cdot\hat{\mathbf{k}}\partial_k\tilde{f}_a^{(0)}(\mathbf{k}) = \frac{2}{\hbar}\int \mathrm{d}^3r\int \frac{\mathrm{d}^3k_1}{(2\pi\hbar)^3}\sin\left(\frac{\mathbf{r}\cdot(\mathbf{k}-\mathbf{k}_1)}{\hbar}\right)V^{\mathrm{eff}}\left(\mathbf{R}+\frac{\mathbf{r}}{2},t\right)\tilde{f}_a^{(0)}(\mathbf{k}_1,\mathbf{R},t). \tag{49}$$

In the non-degenerate limit this is the Wigner form for the off diagonal Kelbg quantum potential [32]. Eq. (48) provides the basis for more general representations in non-perturbative and far from equilibrium states. However, the explicit form for these non-local quantum effects will not be considered further here.

We consider now the collision integrals of (46). Performing the gauge-invariant Fourier transform on the r.h.s. of the kinetic equation which contains all two-particle quantities related to mean field and correlation effects in local approximation – analogously to the homogeneous case [19,20] with $\mathbf{R}$ being an additional parameter – one sees that the Hartree–Fock-contribution vanishes and the collision integral is of the form

$$\widetilde{I}_a(\mathbf{k},\mathbf{R},t) = \pm 2\mathrm{Re}\left[\int_{t_0}^{t} \mathrm{d}\bar{t}\left\{\widetilde{\Sigma}_a^{>}(\mathbf{k}_a^A,\mathbf{R};t,\bar{t})\tilde{g}_a^{<}(\mathbf{k}_a^A,\mathbf{R};\bar{t},t) - \widetilde{\Sigma}_a^{<}(\mathbf{k}_a^A,\mathbf{R};t,\bar{t})\tilde{g}_a^{>}(\mathbf{k}_a^A,\mathbf{R};\bar{t},t)\right\}\right], \tag{50}$$

where the superscript "A" denotes that all momenta are shifted, according to $\mathbf{k}_a^A \equiv \mathbf{k}_a + \mathbf{K}_a^A(t,\bar{t})$, etc. The local approximation can be used if the characteristic length scales of the scattering processes are small compared to the characteristic scale of the confinement potential. Otherwise gradient corrections have to be taken into account [23]. Note that the momentum arguments in all functions are shifted by the field-dependent momentum $\mathbf{K}^A$ which reflects the explicit field-dependence of the two-particle scattering process. This leads to the so-called intracollisional field effect and to nonlinear phenomena including collisional harmonics generation (inverse) bremsstrahlung, multiphoton excitation and ionization, etc. [19,20].

Collecting all terms together, we obtain the kinetic equation in local approximation

$$\begin{aligned}&\left\{\frac{\partial}{\partial t} + \frac{\mathbf{k}}{m_a}\nabla_{\mathbf{R}} - \left(\nabla_{\mathbf{R}} \mathscr{V}_{\mathrm{Q}}^{\mathrm{eff}}(\mathbf{R},\mathbf{k},t) - e_a\mathbf{E}(t)\right)\nabla_{\mathbf{k}}\right\}\tilde{f}_a(\mathbf{k},\mathbf{R},t)\\ &\quad = \pm 2\mathrm{Re}\left[\int_{t_0}^{t} \mathrm{d}\bar{t}\left\{\widetilde{\Sigma}_a^{>}(\mathbf{k}_a^A,\mathbf{R},t,\bar{t})\tilde{g}_a^{<}(\mathbf{k}_a^A,\mathbf{R},\bar{t},t) - \widetilde{\Sigma}_a^{<}(\mathbf{k}_a^A,\mathbf{R},t,\bar{t})\tilde{g}_a^{>}(\mathbf{k}_a^A,\mathbf{R},\bar{t},t)\right\}\right].\end{aligned} \tag{51}$$

We emphasize that this equation is very general. It applies to arbitrary nonequilibrium situations in strong electromagnetic fields with arbitrary amplitude and time-dependence and a weakly inhomogeneous confinement potential. A particular collision process is specified by the appropriate choice of the self-energies. Due to the gauge-invariant derivation, the resulting kinetic equation is gauge-invariant as well.

4.1. Collision integral

In this section, we analyze the collision integrals in the kinetic equation (51) for the case of Coulomb scattering. The main goal here is to see the effects of the electric field, the confinement potential and primarily of the quantum potential on the collision process.

To obtain a closed equation for the Wigner distribution the two-time correlation functions have to be expressed by the distribution function via the so-called reconstruction ansatz. Its original form due to Baym and Kadanoff [15] was generalized by Lipavsky et al. [18] to nonequilibrium systems, properly accounting for causality and retardation effects. The latter ansatz was further generalized to a gauge-invariant form including strong electromagnetic fields [19,20]. Here we account, in addition, for a weakly inhomogeneous confinement potential:

$$\begin{aligned}\tilde{g}_a^{>}(\mathbf{k},\mathbf{R},t_1,t_1') &= \left\{\tilde{g}_a^{\mathrm{R}}(\mathbf{k},\mathbf{R},t_1,t_1')\tilde{f}_a^{>}(\mathbf{k}-\mathbf{K}_a^{\mathrm{A}}(t_1',t_1),\mathbf{R};t_1') - \tilde{f}_a^{>}(\mathbf{k}-\mathbf{K}_a^{\mathrm{A}}(t_1,t_1'),\mathbf{R};t_1)\tilde{g}_a^{\mathrm{A}}(\mathbf{k},\mathbf{R},t_1,t_1')\right\},\\ \tilde{g}_a^{<}(\mathbf{k},\mathbf{R},t_1,t_1') &= \pm\left\{\tilde{g}_a^{\mathrm{R}}(\mathbf{k},\mathbf{R},t_1,t_1')\tilde{f}_a^{<}(\mathbf{k}-\mathbf{K}_a^{\mathrm{A}}(t_1',t_1),\mathbf{R};t_1') - \tilde{f}_a^{<}(\mathbf{k}-\mathbf{K}_a^{\mathrm{A}}(t_1,t_1'),\mathbf{R};t_1)\,\tilde{g}_a^{\mathrm{A}}(\mathbf{k},\mathbf{R},t_1,t_1')\right\},\end{aligned} \tag{52}$$

where $f^{>} \equiv 1 \pm f$ and $f^{<} \equiv f$.

It is reasonable to start with the simplest approximation for Coulomb scattering – the static second Born approximation. The gauge-invariant expression for the self-energy in local approximation is given by [20]

$$\widetilde{\Sigma}_a^{\gtrless}(\mathbf{k}_a,\mathbf{R},t,t') = \mp\frac{\hbar^2}{(2\pi\hbar)^6}\sum_b\int d\mathbf{k}_b d\bar{\mathbf{k}}_a\, d\bar{\mathbf{k}}_b \\ |\,V(\mathbf{k}_a-\bar{\mathbf{k}}_a)|^2\delta(\mathbf{k}_a+\mathbf{k}_b-\bar{\mathbf{k}}_a-\bar{\mathbf{k}}_b)\tilde{g}_a^{\gtrless}(\bar{\mathbf{k}}_b,\mathbf{R},t,t')\tilde{g}_b^{\gtrless}(\bar{\mathbf{k}}_b,\mathbf{R},t,t')\tilde{g}_a^{\lessgtr}(\mathbf{k}_b,\mathbf{R},t',t), \quad (53)$$

which, using (50), leads to the collision integral

$$\widetilde{I}_a(\mathbf{k}_a,\mathbf{R},t) = -2\mathrm{Re}\left[\frac{\hbar^2}{(2\pi\hbar)^6}\sum_b\int d\mathbf{k}_b\, d\bar{\mathbf{k}}_a\, d\bar{\mathbf{k}}_b\,|\,V(\mathbf{k}_a-\bar{\mathbf{k}}_a)|^2\delta(\mathbf{k}_a+\mathbf{k}_b-\bar{\mathbf{k}}_a-\bar{\mathbf{k}}_b)\right. \\ \left.\times\int_{t_0}^{t}\mathrm{d}\bar{t}\{\tilde{g}_a^{>}(\bar{\mathbf{k}}_a^A,\mathbf{R},t,\bar{t})\tilde{g}_b^{>}(\bar{\mathbf{k}}_b^A,\mathbf{R},t,\bar{t})\tilde{g}_b^{<}(\mathbf{k}_b^A,\mathbf{R},\bar{t},t)\tilde{g}_a^{<}(\mathbf{k}_a^A,\mathbf{R},\bar{t},t)-(>\leftrightarrow<)\}\right]. \quad (54)$$

where $\mathbf{k}_a^A(t,\bar{t}) = \mathbf{k}_a + \mathbf{K}^A(t,\bar{t})$. Now we express the two-time correlation functions by the Wigner distributions using the generalized Kadanoff–Baym ansatz (52)

$$\widetilde{I}_a(\mathbf{k}_a,\mathbf{R},t) = -2\frac{\hbar^2}{(2\pi\hbar)^6}\mathrm{Re}\int d\mathbf{k}_b\, d\bar{\mathbf{k}}_a d\bar{\mathbf{k}}_b\,|\,V(\mathbf{k}_a-\bar{\mathbf{k}}_a)|^2\delta(\mathbf{k}_a+\mathbf{k}_b-\bar{\mathbf{k}}_a-\bar{\mathbf{k}}_b) \\ \int_{t_0}^{t}\mathrm{d}\bar{t}\;\tilde{g}_a^R(\bar{\mathbf{k}}_a^A,\mathbf{R},t,\bar{t})\tilde{g}_b^R(\bar{\mathbf{k}}_b^A,\mathbf{R},t,\bar{t})\tilde{g}_b^A(\mathbf{k}_b^A,\mathbf{R},\bar{t},t)\tilde{g}_a^A(\mathbf{k}_a^A,\mathbf{R},\bar{t},t) \\ \left\{\tilde{f}_a^{>}(\bar{\mathbf{k}}_a^Q,\mathbf{R},\bar{t})\tilde{f}_b^{>}(\bar{\mathbf{k}}_b^Q,\mathbf{R},\bar{t})\tilde{f}_b^{<}(\mathbf{k}_b^Q,\mathbf{R},\bar{t})\tilde{f}_a^{<}(\mathbf{k}_a^Q,\mathbf{R},\bar{t})-(>\leftrightarrow<)\right\}, \quad (55)$$

where the shift of the momentum arguments in the distribution functions is now given by $\mathbf{k}_a^Q \equiv \mathbf{k}_a + \mathbf{Q}_a(t,\bar{t})$ with $\mathbf{Q}_a^A(t,t') = \mathbf{K}_a^A(t,t') - \mathbf{K}_a^A(t',t)$. What is left now is to evaluate the spectral information of the four propagators which determines the energy balance of the scattering event in the combined external and confinement fields. Therefore, using the propagator from the approximate spectral function (30), which includes quantum effects via the quantum potential

$$\tilde{g}^{\mathrm{R/A}}(\mathbf{k},\mathbf{R},t,\bar{t}) = \mp\frac{\mathrm{i}}{\hbar}\Theta[\pm(t_1-t_1')]\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\left(\left(\frac{k^2}{2m_a}+V^{\mathrm{eff}}(\mathbf{R},t)\right)\tau+X(\mathbf{k},\mathbf{R},t;\mathbf{E})\tau^3+Y(\mathbf{R},t;\mathbf{E})\tau^5\right)}. \quad (56)$$

Here $\tau = t-\bar{t}$, and $T = (t+\bar{t})/2$. The result for the product of two propagators which has to be computed in the collision term (55) is

$$\tilde{g}_a^A(\mathbf{k}_a^A,\mathbf{R},\bar{t},t)\tilde{g}_a^R(\bar{\mathbf{k}}_a^A,\mathbf{R},t,\bar{t})\tilde{g}_a^A(\mathbf{k}_a^A;\bar{t},t)\tilde{g}_a^R(\bar{\mathbf{k}}_a^A;t,\bar{t}) \to \frac{1}{\hbar^2}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\tau(\varepsilon_a(\mathbf{k}_a,\mathbf{R},\mathbf{E};\tau)-\varepsilon_a(\bar{\mathbf{k}}_a,\mathbf{R},\mathbf{E};\tau))}, \quad (57)$$

$$\varepsilon_a(\mathbf{k}_a,\mathbf{R},\mathbf{E};\tau) \equiv \frac{(k_a^A(\tau))^2}{2m_a}\left(1+\tau^2\frac{1}{6m_a}(\widehat{\mathbf{k}}^A(\tau)\cdot\nabla_R)^2V^{\mathrm{eff}}(\mathbf{R},t)\right), \quad (58)$$

and analogously for particle b. In Eq. (58) $\widehat{\mathbf{k}}_a^A$ denotes a unit vector in the direction of $\mathbf{k}_a^A$

With these propagators the collision term including quantum confinement effects becomes

$$\widetilde{I}_{ab}(\mathbf{k}_a,\mathbf{R},t) = \frac{2}{(2\pi\hbar)^6}\frac{1}{\hbar^2}\int\mathrm{d}\bar{\mathbf{k}}_a\mathrm{d}\mathbf{k}_b\mathrm{d}\bar{\mathbf{k}}_b|V_{ab}(\mathbf{k}_a-\bar{\mathbf{k}}_a)|^2\delta(\mathbf{k}_a+\mathbf{k}_b-\bar{\mathbf{k}}_a-\bar{\mathbf{k}}_b) \\ \int_{t_0}^{t}\mathrm{d}\bar{t}\cos\left\{\frac{\tau}{\hbar}[\varepsilon_a(\mathbf{k}_a,\mathbf{R},\mathbf{E};\tau)+\varepsilon_b(\mathbf{k}_b,\mathbf{R},\mathbf{E};\tau)-\varepsilon_a(\bar{\mathbf{k}}_a,\mathbf{R},\mathbf{E};\tau)-\varepsilon_b(\bar{\mathbf{k}}_b,\mathbf{R},\mathbf{E};\tau)]\right\} \\ \left\{f^{<}(\bar{\mathbf{k}}_a^Q,\mathbf{R},\bar{t})f^{<}(\bar{\mathbf{k}}_b^Q,\mathbf{R},\bar{t})f^{>}(\mathbf{k}_b^Q,\mathbf{R},\bar{t})f^{>}(\mathbf{k}_a^Q,\mathbf{R},\bar{t})-(>\leftrightarrow<)\right\}. \quad (59)$$

As in the homogeneous case, the field drops out of the energy balance of scattering of particles with same charge to mass ratio since the field does not change their distance. In contrast, in the case of electron ion scattering the field changes the energy balance; see [19].

We see from Eq. (59) that the confinement potential does in fact have an influence on the scattering process. The collision integral contains, in addition to the difference of kinetic energies of the particle pair, a term proportional to the local curvature of the effective confinement potential in (58)

The effect of the quantum potential on the scattering process can be analyzed more closely in the field free case. In the limit $\mathbf{E} \to 0$ we can evaluate the integral over time in the Markov limit [14]. Neglecting the time dependence of the distribution functions compared to the correlation time and extending the $\bar{t}$-integration to infinity, the integration can be performed using the integral representation of the airy function. As a result the collision integral becomes

$$I_{ab}(\mathbf{k}_a,\mathbf{R},t) = \frac{2}{(2\pi\hbar)^6}\frac{\pi}{\hbar}\int \mathrm{d}\bar{\mathbf{k}}_a\mathrm{d}\mathbf{k}_b\mathrm{d}\bar{\mathbf{k}}_b |V_{ab}(\mathbf{k}_a-\bar{\mathbf{k}}_a)|^2\delta(\mathbf{k}_a+\mathbf{k}_b-\bar{\mathbf{k}}_a-\bar{\mathbf{k}}_b) \times \frac{1}{\alpha_{ab}}\mathrm{Ai}\left(\frac{\Delta\epsilon_{ab}}{\alpha_{ab}}\right)\left\{f^<(\bar{\mathbf{k}}_a,\mathbf{R},\bar{t})f^<(\bar{\mathbf{k}}_b,\mathbf{R},\bar{t})f^>(\mathbf{k}_b,\mathbf{R},\bar{t})f^>(\mathbf{k}_a,\mathbf{R},\bar{t}) - (>\leftrightarrow<)\right\}, \tag{60}$$

with

$$\alpha_{ab} = \left[\frac{\hbar^2}{4}\left(\left[\frac{\epsilon_a}{m_a}(\hat{\mathbf{k}}_a\cdot\nabla_R)^2 - \frac{\bar{\epsilon}_a}{m_a}(\hat{\bar{\mathbf{k}}}_a\cdot\nabla_R)^2\right]V_a(\mathbf{R},t) + (a\to b)\right)\right]^{1/3},$$

and $\Delta\epsilon_{ab} = \epsilon_a + \epsilon_b - (\bar{\epsilon}_a + \bar{\epsilon}_b)$. The Airy function is peaked at $\Delta\epsilon_{ab} = 0$, i.e. the dominant spectral weight falls on scattering processes which conserve the kinetic energy of the particle pair, as in the case of a homogeneous system (in the Markov limit). The latter case is recovered by the limit $\nabla V(\mathbf{R},t) \to 0$ using the relation $\lim_{\epsilon\to 0}\frac{1}{\epsilon}\mathrm{Ai}\left(\frac{x}{\epsilon}\right) = \delta(x)$, leading to the well-known classical result

$$I_{ab}(\mathbf{k}_a,t) = \frac{2}{(2\pi\hbar)^6}\frac{\pi}{\hbar}\int \mathrm{d}\bar{\mathbf{k}}_a\mathrm{d}\mathbf{k}_b\mathrm{d}\bar{\mathbf{k}}_b |V_{ab}(\mathbf{k}_a-\bar{\mathbf{k}}_a)|^2\delta(\mathbf{k}_a+\mathbf{k}_b-\bar{\mathbf{k}}_a-\bar{\mathbf{k}}_b)\delta(\epsilon_a+\epsilon_b-\bar{\epsilon}_a-\bar{\epsilon}_a) \left\{f^<(\bar{\mathbf{k}}_a,\bar{t})f^<(\bar{\mathbf{k}}_b,\bar{t})f^>(\mathbf{k}_b,\bar{t})f^>(\mathbf{k}_a,\bar{t}) - (>\leftrightarrow<)\right\}. \tag{61}$$

A larger effect of the inhomogeneity on the scattering process occurs on short-time scales of the order of the correlation time where the Markov limit fails, e.g. [14,33]. Then the energy broadening arising from the finite collision duration is additionally increased due to the inhomogeneity of the confinement field, so that collisions that do not conserve the one-particle energy become possible (see Fig. 1).

Aside from Coulomb scattering, there are numerous physical situations where the presence of an inhomogeneous field will have an even more pronounced effect on the microscopic scattering probability. The most important one is inelastic scattering. Indeed if particles, after the collision appear in a different quantum state (energy level or band) with a different dispersion (effective mass), the quantum potential will be different before and after the collision, even if the external field is independent of the quantum state. This effect should be directly observable in confined quantum systems undergoing e.g. collisional excitation or ionization.

5. Discussion

Weak inhomogeneity covers a broad class of many-particle systems of current interest, including electrons in quantum dots, ultracold ions in traps, valence electrons in metal clusters and so on. In this paper, a quantum potential for particle systems in a weakly inhomogeneous confinement potential and a strong electromagnetic field – that means a nonequilibrium quantum potential – has been derived. With this momentum-dependent quantum potential the spectral function for systems of trapped quantum particles in a strong electric field has been calculated, obeying Heisenberg's uncertainty principle. With the spectral function the density of states has been computed, now permitting the existence of negative energy values because of quantum confinement effects. Special attention has been devoted to an investigation of the modification of the collision term in the Boltzmann equation including the quantum potential. Due to quantum confinement effects the sharp peak of the energy balance in the classical case is smeared out. Because of the gauge-invariant expression for the GKBA and the self-energies in Born approximation an analysis of the collision terms including the homogeneous electric field is straight forward.

Acknowledgements

This work is supported by the Innovationsfond Schleswig-Holstein, and by the US Department of Energy Award DE-FG02-07ER54946.

Appendix A. Approximate quantum potential

In this appendix, the derivation of (26) for the quantum potential at $\hbar = 0$ is outlined. Next, its specialization to a slowly varying confinement potential is described, leading to (29).

Eq. (25) at $\hbar = 0$ becomes

$$V_{\mathrm{Q}}^{\mathrm{eff}}(\mathbf{k},\mathbf{R},\tau,t;\hbar=0) = \lim_{\hbar\to 0}\frac{1}{2}\int \mathrm{d}\mathbf{r}\int\frac{\mathrm{d}\mathbf{k}}{(2\pi\hbar)^3}\left\{\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\mathbf{r}(\mathbf{k}-\mathbf{k}_1)}V^{\mathrm{eff}}\left(\mathbf{R}+\frac{\mathbf{r}}{2},t+\frac{\tau}{2}\right)+\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathbf{r}(\mathbf{k}-\mathbf{k}_1)}V^{\mathrm{eff}}\left(\mathbf{R}+\frac{\mathbf{r}}{2},t-\frac{\tau}{2}\right)\right\}$$
$$\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}(\phi(\mathbf{k}_1,\mathbf{R},\tau,t)-\phi(\mathbf{k},\mathbf{R},\tau,t))}+\frac{1}{8m_a}\left(\int_0^\tau \mathrm{d}\tau'\nabla_R V_{\mathrm{Q}}^{\mathrm{eff}}(\mathbf{k}+\kappa(\tau,t),\mathbf{R},\tau',t;\hbar=0)\right)^2$$
$$+\frac{1}{2m_a}\mathbf{K}^A_{a-}(t_1,t_1')\cdot\int_0^\tau \mathrm{d}\tau'\nabla_R V_{\mathrm{Q}}^{\mathrm{eff}}(\mathbf{k}+\kappa(\tau,t),\mathbf{R},\tau',t;\hbar=0), \tag{A1}$$

where for notational simplicity $\phi(\mathbf{k},\mathbf{R},\tau,t)$ has been introduced

$$\phi(\mathbf{k},\mathbf{R},\tau,t)\equiv\frac{\tau}{2m_a}(k^2)+\int_0^\tau \mathrm{d}\tau' V_{\mathrm{Q}}^{\mathrm{eff}}(\mathbf{k}+\kappa(\tau,t),\mathbf{R},\tau',t). \tag{A2}$$

The limit $\hbar = 0$ in the first term requires some care of (A1). First write

$$\int \mathrm{d}\mathbf{r}\ \mathrm{e}^{\pm\frac{\mathrm{i}}{\hbar}\mathbf{r}\cdot(\mathbf{k}_1-\mathbf{k})}V^{\mathrm{eff}}\left(\mathbf{R}+\frac{\mathbf{r}}{2},t\pm\frac{\tau}{2}\right)=\int \mathrm{d}\mathbf{r}\ \mathrm{e}^{\pm\frac{\mathrm{i}}{\hbar}\mathbf{r}\cdot(\mathbf{k}_1-\mathbf{k})}\mathrm{e}^{\frac{\mathbf{r}}{2}\nabla_R}V^{\mathrm{eff}}\left(\mathbf{R},t\pm\frac{\tau}{2}\right)$$
$$=\mathrm{e}^{\pm\frac{\hbar}{\mathrm{i}}\nabla_{k_1}\cdot\nabla_R}\int \mathrm{d}\mathbf{r}\ \mathrm{e}^{\pm\frac{\mathrm{i}}{\hbar}\mathbf{r}\cdot(\mathbf{k}_1-\mathbf{k})}V^{\mathrm{eff}}\left(\mathbf{R},t\pm\frac{\tau}{2}\right)$$
$$=\mathrm{e}^{\pm\frac{\hbar}{\mathrm{i}}\nabla_{k_1}\cdot\nabla_R}V^{\mathrm{eff}}\left(\mathbf{R},t\pm\frac{\tau}{2}\right)(2\pi\hbar)^3\delta(\mathbf{k}_1-\mathbf{k}). \tag{A3}$$

Next, perform the $\mathbf{k}$ integration

$$\lim\int\frac{\mathrm{d}\mathbf{k}}{(2\pi\hbar)^3}\int \mathrm{d}\mathbf{r}\ \mathrm{e}^{\pm\frac{\mathrm{i}}{\hbar}\mathbf{r}\cdot(\mathbf{k}_1-\mathbf{k})}V^{\mathrm{eff}}\left(\mathbf{R}\pm\frac{\mathbf{r}}{2},t\pm\frac{\tau}{2}\right)\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}(\phi(\mathbf{k}_1,\mathbf{R},\tau,t)-\phi(\mathbf{k},\mathbf{R},\tau,t))}$$
$$=\lim\int \mathrm{d}\mathbf{k}\ \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}(\phi(\mathbf{k}_1,\mathbf{R},\tau,t)-\phi(\mathbf{k},\mathbf{R},\tau,t))}\mathrm{e}^{\pm\frac{\hbar}{\mathrm{i}}\nabla_{k_1}\cdot\nabla_R}V^{\mathrm{eff}}\left(\mathbf{R},t\pm\frac{\tau}{2}\right)\delta(\mathbf{k}_1-\mathbf{k})$$
$$=\lim\sum_{n=0}^{\infty}\frac{1}{n!}\int \mathrm{d}\mathbf{k}\ \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}(\phi(\mathbf{k}_1,\mathbf{R},\tau,t)-\phi(\mathbf{k},\mathbf{R},\tau,t))}\left(\pm\frac{\hbar}{\mathrm{i}}\nabla_{k_1}\cdot\nabla_R\right)^n V^{\mathrm{eff}}\left(\mathbf{R},t\pm\frac{\tau}{2}\right)\delta(\mathbf{k}_1-\mathbf{k})$$
$$=\lim\int \mathrm{d}\mathbf{k}\ \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}(\phi(\mathbf{k}_1,\mathbf{R},\tau,t)-\phi(\mathbf{k},\mathbf{R},\tau,t))}\delta(\mathbf{k}_1-\mathbf{k})\sum_{n=0}^{\infty}\frac{1}{n!}(\pm\nabla_{k_1}\phi(\mathbf{k}_1,\mathbf{R},\tau,t)\cdot\nabla_R)^n V^{\mathrm{eff}}\left(\mathbf{R},t\pm\frac{\tau}{2}\right)$$
$$+\text{order }\hbar. \tag{A4}$$

In the last line integrations by parts have been performed, and it has been recognized that terms with second or higher derivatives of ϕ are at least of order $\hbar$ and vanish in the limit indicated. Also, to combine the powers of $\nabla_{k_1}\phi$ and ∇_R it must be understood that ∇_R is taken at constant ϕ. Summing the series gives

$$\lim\int\frac{\mathrm{d}\mathbf{k}}{(2\pi\hbar)^3}\int \mathrm{d}\mathbf{r}\ \mathrm{e}^{\pm\frac{\mathrm{i}}{\hbar}\mathbf{r}\cdot(\mathbf{k}_1-\mathbf{k})}V^{\mathrm{eff}}\left(\mathbf{R}\pm\frac{\mathbf{r}}{2},t\pm\frac{\tau}{2}\right)\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}(\phi(\mathbf{k}_1,\mathbf{R},\tau,t)-\phi(\mathbf{k},\mathbf{R},\tau,t))}$$
$$=\mathrm{e}^{\pm\nabla_{k_1}\phi\cdot\nabla_R}V^{\mathrm{eff}}\left(\mathbf{R},t\pm\frac{\tau}{2}\right)=V^{\mathrm{eff}}\left(\mathbf{R}\pm\nabla_k\phi,t\pm\frac{\tau}{2}\right). \tag{A5}$$

Finally, (A1) becomes (26) of the text

$$V_{\mathrm{Q}}^{\mathrm{eff}}(\mathbf{k},\mathbf{R},\tau,t;\hbar=0)=\frac{1}{2}\left(V^{\mathrm{eff}}\left(\mathbf{R}+\nabla_k\phi,t+\frac{\tau}{2}\right)+V^{\mathrm{eff}}\left(\mathbf{R}-\nabla_k\phi,t-\frac{\tau}{2}\right)\right)+\frac{1}{8m_a}\left(\int_0^\tau \mathrm{d}\tau'\nabla_R V_{\mathrm{Q}}^{\mathrm{eff}}(\mathbf{k}+\kappa(\tau,\tau',t),\mathbf{R},\tau',t;\hbar=0)\right)^2+\frac{1}{2m_a}\mathbf{K}_{a-}^{A}(t_1,t_1')\cdot\int_0^\tau \mathrm{d}\tau'\nabla_R V_{\mathrm{Q}}^{\mathrm{eff}}(\mathbf{k}+\kappa(\tau,\tau',t),\mathbf{R},\tau',t;\hbar=0), \tag{A6}$$

where the function $\nabla_k\phi$ is now

$$\nabla_k\phi(\mathbf{k},\mathbf{R},\tau,t)=\tau\frac{\mathbf{k}}{m_a}+\int_0^\tau \mathrm{d}\tau'\nabla_k V_{\mathrm{Q}}^{\mathrm{eff}}(\mathbf{k}+\kappa(\tau,\tau',t),\mathbf{R},\tau',t;\hbar=0). \tag{A7}$$

Note that the equation is real and therefore admits real solutions.

Now specialize to the case in which both the electromagnetic field and the confining potential are slowly varying in time, such that

$$\mathbf{K}_{a-}^{A}(t_1,t_1')\to-\frac{1}{2}e_a\mathbf{E}(t)\tau,\quad \boldsymbol{\kappa}(\tau,\tau',t)\to 0,\quad V^{\mathrm{eff}}\left(\mathbf{R}\pm\nabla_k\phi,t\pm\frac{\tau}{2}\right)\to V^{\mathrm{eff}}(\mathbf{R}\pm\nabla_k\phi,t). \tag{A8}$$

$$V_{\mathrm{Q}}^{\mathrm{eff}}(\mathbf{k},\mathbf{R},\tau,t;\hbar=0)=\frac{1}{2}(V^{\mathrm{eff}}(\mathbf{R}+\nabla_k\phi,t)+V^{\mathrm{eff}}(\mathbf{R}-\nabla_k\phi,t))+\frac{1}{8m_a}\left(\int_0^\tau \mathrm{d}\tau'\nabla_R V_{\mathrm{Q}}^{\mathrm{eff}}(\mathbf{k},\mathbf{R},\tau',t;\hbar=0)\right)^2-\tau\frac{1}{4m_a}e_a\mathbf{E}(t)\cdot\int_0^\tau \mathrm{d}\tau'\nabla_R V_{\mathrm{Q}}^{\mathrm{eff}}(\mathbf{k},\mathbf{R},\tau',t;\hbar=0), \tag{A9}$$

$$\nabla_k\phi(\mathbf{k},\mathbf{R},\tau,t)=\tau\frac{\mathbf{k}}{m_a}+\int_0^\tau \mathrm{d}\tau'\nabla_k V_{\mathrm{Q}}^{\mathrm{eff}}(\mathbf{k},\mathbf{R},\tau',t;\hbar=0). \tag{A10}$$

Clearly, $V_{\mathrm{Q}}^{\mathrm{eff}}\to 0$ as $V^{\mathrm{eff}}\to 0$, so the second and third terms are proportional to gradients of V^{eff}. If this confining potential is slowly varying in space, and expansion in the gradients of V^{eff} is possible. It follows directly that the results to second order in these gradients is

$$V_{\mathrm{Q}}^{\mathrm{eff}}(\mathbf{k},\mathbf{R},\tau,t;\hbar=0)=V^{\mathrm{eff}}(\mathbf{R},t)+\frac{1}{2}\left(\tau\frac{\mathbf{k}}{m_a}\cdot\nabla_R\right)^2V^{\mathrm{eff}}(\mathbf{R},t)+\cdots+\tau^2\frac{1}{8m_a}(\nabla_R V^{\mathrm{eff}}(\mathbf{R},t)+\cdots)^2-\tau^2\frac{1}{4m_a}e_a\mathbf{E}(t)\cdot\nabla_R V^{\mathrm{eff}}(\mathbf{R},t)+\tau^4\frac{1}{3}\left(\frac{e_a}{4m_a}\mathbf{E}(t)\cdot\nabla_R\right)^2V^{\mathrm{eff}}(\mathbf{R},t)+\cdots \tag{A11}$$

This gives (29) of the text.

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