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Ionization Kinetics in Laser Plasmas

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Abstract. The interaction of ultrashort laser pulses with matter is a topic of growing interest. One of the challenging tasks arising in this field is the description of the ionization and recombination kinetics of atoms in a partially ionized plasma under the influence of the laser field. Since the behavior of atoms in plasmas is strongly influenced by the plasma environment, the equation for the two-particle density matrix in extended ladder approximation is an appropriate starting point for our investigations. We show that from this equation, in a properly chosen basis, a coupled system of equations for the distribution functions of atoms and free particles as well as for the transition matrix elements follows. This system is a generalization of the Bloch equations well-known from atomic and semiconductor physics. We discuss the case of strong fields where phenomena like higher harmonics and multi-photon processes occur.

1. Introduction

We consider a partially ionized plasma. Under the influence of an external electromagnetic field, e.g. a strong laser field, very interesting effects may be observed. Multiphoton processes, creation of higher harmonics, nonlinear absorption etc. are typical consequences. The theoretical approach to the processes in strong laser fields requires, of course, nonperturbative treatments with respect to the field.

Of special interest are the ionization of the atoms due to the laser field and the energy transfer between field and plasma. The elementary process of ionization in strong laser fields was investigated for the first time in the pioneering work of Keldysh [1] and further in the important papers of Faisal [2] and Reiss [3]. In these papers, the transition probability from an atomic level into the continuum states was determined in the strong field approximation, i.e. by nonperturbative calculations with respect to the external field. It was shown that, due to the strong field, multiphoton processes occur. In spite of the success of the Keldysh–Faisal–Reiss (KFR) theory, this approach is confronted with several difficulties, see, e.g., [4] and [5]. In particular, the KFR theory is strongly gauge dependent and not invariant by removal of the A^2 contribution to the Hamiltonian which is, in the homogeneous case (dipole approximation), a c–number.

In the present paper we will consider the energy transfer between plasma and field and the kinetics of the laser ionization in the surrounding partially ionized plasma. We start from the equation for the two-particle density matrix in extended binary collision approximation. The external electromagnetic field is introduced by a manifestly gauge invariant minimal coupling substitution $\mathbf{p} \to \mathbf{p} - e\mathbf{A}$. On the basis of this equation we describe the ionization process in the surrounding plasma. We mention that, by this procedure, the A^2 term is cancelled.

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This paper is organized as follows: In the next section, we introduce the basic equations for the description of the matter–field interaction and the two-particle properties of the plasma. Furthermore, we present the appropriate basis and derive the general matrix Bloch equation. Focusing on the description of ionization, a system of equations follows which we call "Plasma Bloch equations" in analogy and segregation to those of atomic and semiconductor physics. They will be derived in Sec. 3. Afterwards we discuss the behavior of the transition function as a central quantity of the formalism for the case of arbitrary fields as well as in the limiting case of weak fields in Sec. 4. In Sec. 5 we investigate the energy transfer between field and plasma by deriving absorption as well as rate coefficients focusing on the strong field case. Finally, we conclude with an outlook to continuative investigations to the topic.

2. Basic equations

We consider the interaction between a partially ionized plasma and the electromagnetic field. This interaction is connected with an energy transfer and a polarization of the plasma. Energy transfer and polarization are determined by the electric current density $\mathbf{j}(\mathbf{E})$ due to the field $\mathbf{E}$ by the relations

$$\frac{dW^{\rm kin}}{dt} + \frac{dW^{\rm pot}}{dt} = \mathbf{j}\cdot\mathbf{E}\,, \quad \frac{d\mathbf{P}}{dt} = \mathbf{j} \tag{1}$$

i.e., the change of the total energy of the system of particles is equal to $\mathbf{j}\cdot\mathbf{E}$ which is in turn the energy loss of the electromagnetic field due to Poynting's theorem. From the quantities $\mathbf{j}\cdot\mathbf{E}$ and $\mathbf{P}$ we get, as well known, further important quantities like the absorption, reflexion, and refraction. The absorption may be connected with the ionization of the atoms in the partially ionized plasma.

A central quantity is, therefore, the electric current density defined by

$$\mathbf{j} = \sum_a \,\mathrm{Tr}\, \frac{e_a\mathbf{p}_a}{m_a} F_a \tag{2}$$

Here, F_a is the single-particle density operator of the species a. For the determination of F_a we start from the first equation of the Bogolyubov hierarchy [6, 7]

$$i\hbar\frac{d}{dt}F_a - \left[H_a + \Sigma_a^{\rm HF}, F_a\right] = \sum_b n_b \mathrm{Tr}_b\left[V_{ab}, g_{ab}\right] \tag{3}$$

Due to the two-particle interaction this equation is not closed, it is coupled with the equation for the two-particle density operator F_{ab}. Therefore, we have to consider the second equation of the hierarchy given by [6, 7]

$$i\hbar\frac{d}{dt}F_{ab} - \left[\tilde{H}^0_{ab} + V_{ab}, F_{ab}\right] = \sum_c n_c \mathrm{Tr}_c\{V_{ac}+V_{bc}, F_{abc}\}\,, \tag{4}$$

where the correlation functions g_{ab} and g_{abc} are introduced by

$$\begin{aligned} F_{ab} &= F_aF_b + g_{ab} \\ F_{abc} &= F_aF_bF_c + F_ag_{bc} + F_bg_{ac} + F_cg_{ab} + g_{abc}. \end{aligned} \tag{5}$$

In this paper we will consider the behavior of a pair of charged particles in a plasma under the influence of an external electromagnetic field. For this purpose, we introduce the electromagnetic field into $\tilde{H}^0_{ab}$ by a manifestly gauge invariant minimal coupling substitution $\mathbf{p}\to\mathbf{p}-e\mathbf{A}$:

$$\tilde{H}^0_{ab} = \frac{1}{2m_a}\left(\mathbf{p}_a - e_a\mathbf{A}\right)^2 + \frac{1}{2m_b}\left(\mathbf{p}_b - e_b\mathbf{A}\right)^2 = H^0_{ab} + U_{ab}, \tag{6}$$

$$H^0_{ab} = \frac{\mathbf{p}_a^2}{2m_a} + \frac{\mathbf{p}_b^2}{2m_b}, \tag{7}$$

$$U_{ab} = -\mathbf{A} \cdot \left(\frac{e_a \mathbf{p}_a}{m_a} + \frac{e_b \mathbf{p}_b}{m_b} \right) + \frac{A^2}{2} \left(\frac{e_a^2}{m_a} + \frac{e_b^2}{m_b} \right). \tag{8}$$

Here, U_{ab} is given in Coulomb gauge, i.e. $\nabla \mathbf{A} = 0$.

It is convenient to introduce relative and center-of-mass coordinates by

$$M\mathbf{p} = m_b \mathbf{p}_a - m_a \mathbf{p}_b, \quad \mathbf{P} = \mathbf{p}_a + \mathbf{p}_b,$$

In these coordinates, the contributions to $\tilde{H}^0_{ab}$ read

$$H^0_{ab} = \frac{\mathbf{P}^2}{2M} + \frac{\mathbf{p}^2}{2\mu}, \tag{9}$$

$$U_{ab} = -\mathbf{A} \cdot \left(\frac{e_t \mathbf{P}}{M} + \frac{e_r \mathbf{p}}{\mu} \right) + \frac{A^2}{2} \left(\frac{e_t^2}{M} + \frac{e_r^2}{\mu} \right), \tag{10}$$

where M and μ are the total and reduced masses,

$$M = m_a + m_b, \quad \frac{1}{\mu} = \frac{1}{m_a} + \frac{1}{m_b},$$

and e_t and e_r the total and reduced charges, respectively,

$$e_t = e_a + e_b, \quad Me_r = m_b e_a - m_a e_b.$$

Obviously, in the case of single ionization, the field affects only the relative motion of electron–ion pairs since, due to $e_a = -e_b = e$, we get immediately $e_r = e$ and $e_t = 0$.

For the decoupling of the equations (3) and (4) from the full Bogolyubov hierarchy we neglect the three-particle correlations in Eq. (4), i.e., we assume $g_{abc} = 0$. Then it is shown in [6] and [7] that Eq. (4) simplifies to the extended binary collision approximation

$$i\hbar \frac{\partial}{\partial t} F_{ab}(t) - \left[H^0_{ab} + V_{ab}, F_{ab}(t) \right] - \left[U_{ab}(t), F_{ab}(t) \right]$$
$$- \left[\left(F_a(t) + F_b(t) \right) V_{ab}, F_{ab}(t) \right] - \left[\left(\Sigma_a^{\mathrm{HF}}(t) + \Sigma_b^{\mathrm{HF}}(t) \right), F_{ab}(t) \right] = 0 \tag{11}$$

which describes the kinetics of the two-particle properties like bound and scattering states in the plasma. In Eq. (11), the first two terms represent the usual binary collision approximation, the third term contains the direct external field contribution, whereas the further terms arise from many-particle effects: Pauli blocking and Hartree–Fock (mean field + exchange) self-energy.

Let us come back to the current density. The balance equation for the electric current density follows in well known manner from the first equation of the hierarchy (3) and reads (after symmetrization with respect to the species a, b)

$$\frac{d\mathbf{j}(t)}{dt} - \sum_a \frac{n_a e_a^2}{m_a} \mathbf{E} = \frac{1}{2i\hbar} \sum_{ab} \int \frac{d\mathbf{p}_a d\mathbf{p}_b}{(2\pi\hbar)^6} \left(\frac{e_a \mathbf{p}_a}{m_a} + \frac{e_b \mathbf{p}_b}{m_b} \right) \langle \mathbf{p}_a \mathbf{p}_b \left| [V_{ab}, F_{ab}(t)] \right| \mathbf{p}_b \mathbf{p}_a \rangle \,. \tag{12}$$

For the further investigation of the kinetics of the partially ionized plasma in interaction with the external electromagnetic field, equation (11) has to be represented in an appropriate basis. With respect to the bound states in such a plasma, we choose a representation where the basis states are given by the bound and scattering eigenstates of the field-free two-particle Hamiltonian,

$$\left(H^0_{ab} + V_{ab}\right) |\mathbf{P}\alpha\rangle \;=\; E_{\alpha\mathbf{P}} \,|\mathbf{P}\alpha\rangle \,, \tag{13}$$

$$|\mathbf{P}\alpha t\rangle \;=\; e^{-\frac{i}{\hbar}E_{\alpha\mathbf{P}}t}\,|\mathbf{P}\alpha\rangle \,. \tag{14}$$

Here, α denotes a complete set of quantum numbers n, l, m in the case of bound states and the relative momentum $\mathbf{p}$ in the case of scattering states, respectively,

$$\alpha = \begin{cases} \{n, l, m\} & \text{for bound states} \\ \mathbf{p} & \text{for scattering states.} \end{cases}$$

In the representation (13), we get for the equation (11)

$$\left[i\hbar\frac{\partial}{\partial t} - (E_{\alpha\mathbf{P}} - E_{\alpha'\mathbf{P}})\right] F^{\alpha\alpha'}_{ab}(\mathbf{P};t) = \sum_{\bar{\alpha}} \left[\hbar\Omega^{\alpha\bar{\alpha}}_{\mathrm{R}}(\mathbf{P};t) F^{\bar{\alpha}\alpha'}_{ab}(\mathbf{P};t) - F^{\alpha\bar{\alpha}}_{ab}(\mathbf{P};t)\hbar\tilde{\Omega}^{\bar{\alpha}\alpha'}_{\mathrm{R}}(\mathbf{P};t)\right], \tag{15}$$

with the matrix elements given by

$$\langle\alpha\mathbf{P}|\, F_{ab}(t)\, |\mathbf{P}'\alpha'\rangle = F^{\alpha\alpha'}_{ab}(\mathbf{P};t)(2\pi\hbar)^3\delta(\mathbf{P} - \mathbf{P}'), \tag{16}$$

$$\begin{aligned}\langle\alpha\mathbf{P}|\, U_{ab}(t) + \Sigma^{\mathrm{HF}}_a(t) + \Sigma^{\mathrm{HF}}_b(t) \pm [F_a(t) + F_b(t)]\, V_{ab}\, |\mathbf{P}'\alpha'\rangle& \\ = \hbar\Omega^{\alpha\alpha'}_{\mathrm{R}}(\mathbf{P};t)(2\pi\hbar)^3\delta(\mathbf{P} - \mathbf{P}')&,\end{aligned} \tag{17}$$

$$\begin{aligned}\langle\alpha\mathbf{P}|\, U_{ab}(t) + \Sigma^{\mathrm{HF}}_a(t) + \Sigma^{\mathrm{HF}}_b(t) \pm V_{ab}\, [F_a(t) + F_b(t)]\, |\mathbf{P}'\alpha'\rangle& \\ = \hbar\tilde{\Omega}^{\alpha\alpha'}_{\mathrm{R}}(\mathbf{P};t)(2\pi\hbar)^3\delta(\mathbf{P} - \mathbf{P}')&.\end{aligned} \tag{18}$$

Here, $\hbar\Omega_{\mathrm{R}}$ is an abbreviation which can be interpreted as a generalized Rabi energy, i.e., the field contribution renormalized by Hartree–Fock and Pauli blocking terms. Notice that U_{ab} in Eqs. (17,18) does not contain the term quadratic in A (cf. Eq. (10)), because it cancels out in the homogeneous case in the commutator. The quantities defined in (17,18) are related to each other by $\left[\hbar\Omega^{\alpha\alpha'}_{\mathrm{R}}(\mathbf{P};t)\right]^* = \hbar\tilde{\Omega}^{\alpha'\alpha}_{\mathrm{R}}(\mathbf{P};t)$.

Thus, the representation chosen above leads to a transformation of (11) into a matrix equation. Due to the inclusion of the scattering continuum, F_{ab} is, in principle, a matrix of infinite rank. If there are $N - 1$ bound states, we obtain N^2 coupled equations. Even if one makes use of the symmetry relation $\left[F^{\alpha\alpha'}_{ab}(\mathbf{P};t)\right]^* = F^{\alpha'\alpha}_{ab}(\mathbf{P};t)$, there remain $\frac{N}{2}(N+1)$ equations.

Obviously, the diagonal elements $F^{\alpha\alpha}_{ab}$ are related to the occupation numbers of the respective state. Due to the external field we have nonvanishing nondiagonal elements $F^{\alpha\alpha'}_{ab}$. They are connected with transitions between two states α and α'. Here, we have to consider several situations:

(i) If both α and α' denote bound states, the equations describe excitation and deexcitation processes of atoms in the plasma. Then we recover the familiar atomic Bloch equations [8, 9, 10].

(ii) If both α and α' denote continuum (scattering) states, processes like (inverse) bremsstrahlung are described.

(iii) In the case most interesting for our investigations, however, α denotes a bound state and α' a scattering state (or vice versa). The transitions between them, i.e. ionization and recombination processes, are thus included in Eq. (15).

3. Plasma Bloch equations

3.1. Derivation of the plasma Bloch equations

Since we are especially interested in the ionization kinetics, in the following, we do not include (de)excitation and (de)acceleration processes, i.e., transitions between discrete states and transitions within the scattering continuum. Therefore, we consider a system with one single bound state and denote

$$\alpha = \begin{cases} b & \text{"bound"} \quad \text{bound state} \\ f & \text{"free"} \quad \text{scattering state.} \end{cases}$$

Obviously, for that system only the physics beyond the familiar atomic Bloch equations remains from Eq. (15). We get a system of three coupled equations:

$$i\hbar\frac{\partial}{\partial t}F_{bb}(\mathbf{P};t) = 2i\int\frac{d\bar{\mathbf{p}}}{(2\pi\hbar)^3}\,\mathrm{Im}\left\{\hbar\Omega_{\mathrm{R}}^{bf}(\mathbf{P},\bar{\mathbf{p}};t)F_{fb}(\mathbf{P},\bar{\mathbf{p}};t)\right\}, \tag{19}$$

$$\begin{aligned}&\left[i\hbar\frac{\partial}{\partial t} - (E(\mathbf{p}) - E(\mathbf{p}'))\right]F_{ff}(\mathbf{P},\mathbf{p},\mathbf{p}';t)\\ &-\int\frac{d\bar{\mathbf{p}}}{(2\pi\hbar)^3}\left[\hbar\Omega_{\mathrm{R}}^{ff}(\mathbf{P},\mathbf{p},\bar{\mathbf{p}};t)F_{ff}(\mathbf{P},\bar{\mathbf{p}},\mathbf{p}';t) - F_{ff}(\mathbf{P},\mathbf{p},\bar{\mathbf{p}};t)\hbar\tilde{\Omega}_{\mathrm{R}}^{ff}(\mathbf{P},\bar{\mathbf{p}},\mathbf{p}';t)\right]\\ &= \hbar\Omega_{\mathrm{R}}^{fb}(\mathbf{P},\mathbf{p};t)F_{bf}(\mathbf{P},\mathbf{p}';t) - F_{fb}(\mathbf{P},\mathbf{p};t)\hbar\tilde{\Omega}_{\mathrm{R}}^{bf}(\mathbf{P},\mathbf{p}';t),\end{aligned} \tag{20}$$

$$\begin{aligned}&\left[i\hbar\frac{\partial}{\partial t} - \left(E(\mathbf{p}) - \tilde{E}_b\right)\right]F_{fb}(\mathbf{P},\mathbf{p};t) - \int\frac{d\bar{\mathbf{p}}}{(2\pi\hbar)^3}\hbar\Omega_{\mathrm{R}}^{ff}(\mathbf{P},\mathbf{p},\bar{\mathbf{p}};t)F_{fb}(\mathbf{P},\bar{\mathbf{p}};t)\\ &= \hbar\Omega_{\mathrm{R}}^{fb}(\mathbf{P},\mathbf{p};t)F_{bb}(\mathbf{P};t) - \int\frac{d\bar{\mathbf{p}}}{(2\pi\hbar)^3}F_{ff}(\mathbf{P},\mathbf{p},\bar{\mathbf{p}};t)\hbar\tilde{\Omega}_{\mathrm{R}}^{fb}(\mathbf{P},\bar{\mathbf{p}};t),\end{aligned} \tag{21}$$

$$F_{bf}(\mathbf{P},\mathbf{p};t) = [F_{fb}(\mathbf{P},\mathbf{p};t)]^*, \tag{22}$$

where $\tilde{E}_b$ denotes the binding energy renormalized by Hartree–Fock and Pauli blocking contributions,

$$\begin{aligned}\tilde{E}_b &= E_b + \int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\,|\varphi_b(\mathbf{p})|^2\left\{\Sigma_1^{\mathrm{HF}}(\mathbf{P},\mathbf{p};t) + \Sigma_2^{\mathrm{HF}}(\mathbf{P},\mathbf{p};t)\right.\\ &\qquad\left.\mp[E(\mathbf{p}) - E_b]\,[F_1(\mathbf{P},\mathbf{p};t) + F_2(\mathbf{P},\mathbf{p};t)]\right\}\end{aligned} \tag{23}$$

with $F_{1/2}(\mathbf{P},\mathbf{p};t) = F_{1/2}\left(\frac{m_{1/2}}{M}\mathbf{P}\pm\mathbf{p};t\right)$ and $\Sigma_{1/2}^{\mathrm{HF}}(\mathbf{P},\mathbf{p};t) = \Sigma_{1/2}^{\mathrm{HF}}\left(\frac{m_{1/2}}{M}\mathbf{P}\pm\mathbf{p};t\right)$. Here, $\varphi_b(\mathbf{p}) = \langle\mathbf{p}|n\rangle$ being the Fourier transform of the bound state wave function.

Keeping in mind the relation (22), we have obtained a system of three equations for three unknown functions the physical meaning of which is quite obvious. F_{bb} is, in principle, equivalent to the distribution function of the atoms in the plasma. F_{ff} is the binary distribution of the unbound electron–ion pairs. From the system (19–22) we see that the dynamics of both distributions is driven by the function F_{fb} describing the transition between bound and continuum states. The latter quantity is closely connected with the polarization function. Its time evolution, in turn, is determined by the distributions. Notice that, even if we neglect three-particle collisions, ionization and recombination takes place due to the presence of the external field contained in the Rabi energies $\hbar\Omega_{\mathrm{R}}$.

3.2. Field terms. Renormalized Rabi energies
As we have seen before, the matrix elements of the field contribution to the Hamiltonian U_{ab} can be represented in the form of Rabi energies $\hbar\Omega_{\mathrm{R}}$. Since we consider electron–ion pairs, U_{ab} from Eq. (10) simplifies to

$$U_{ab} = -\frac{e}{\mu}\mathbf{A}\cdot\mathbf{p}.$$

Thus, the Rabi energies are given by

$$\hbar\Omega_{\mathrm{R}}^{ff}(\mathbf{P},\mathbf{p},\mathbf{p}';t) = -\frac{e}{\mu}\mathbf{A}(t)\cdot\langle+\mathbf{p}|\mathbf{p}|\mathbf{p}'+\rangle + \int\frac{d\bar{\mathbf{p}}}{(2\pi\hbar)^3}\varphi^*_{\mathbf{p}+}(\bar{\mathbf{p}})\varphi_{\mathbf{p}'+}(\bar{\mathbf{p}})$$
$$\times\left\{\Sigma_1^{\mathrm{HF}}(\mathbf{P},\bar{\mathbf{p}};t)+\Sigma_2^{\mathrm{HF}}(\mathbf{P},\bar{\mathbf{p}};t)\mp[E(\bar{\mathbf{p}})-E(\mathbf{p})]\,[F_1(\mathbf{P},\bar{\mathbf{p}};t)+F_2(\mathbf{P},\bar{\mathbf{p}};t)]\right\}, \qquad (24)$$

$$\hbar\Omega_{\mathrm{R}}^{fb}(\mathbf{P},\mathbf{p};t) = -\frac{e}{\mu}\mathbf{A}(t)\cdot\langle+\mathbf{p}|\,\mathbf{p}\,|n\rangle + \int\frac{d\bar{\mathbf{p}}}{(2\pi\hbar)^3}\varphi^*_{\mathbf{p}+}(\bar{\mathbf{p}})\varphi_b(\bar{\mathbf{p}})$$
$$\times\left\{\Sigma_1^{\mathrm{HF}}(\mathbf{P},\bar{\mathbf{p}};t)+\Sigma_2^{\mathrm{HF}}(\mathbf{P},\bar{\mathbf{p}};t)\mp[E(\bar{\mathbf{p}})-E(\mathbf{p})]\,[F_1(\mathbf{P},\bar{\mathbf{p}};t)+F_2(\mathbf{P},\bar{\mathbf{p}};t)]\right\} \qquad (25)$$

with $\varphi_{\mathbf{p}+}(\mathbf{p})$ being the Fourier transform of the scattering wave function. These relations show that the bare field terms are renormalized by many-particle effects in form of Hartree–Fock energies and Pauli blocking contributions.

4. Analysis of the transition function

4.1. Strong fields. Higher harmonics. Multi-photon processes
In order to investigate the full ionization kinetics, the system (19–22) has to be solved selfconsistently. This is, of course, a very complicated task especially due to the momentum dependences of the quantities which is, up to now, numerically not feasible. Therefore, we approach the problem by analyzing limiting cases which simplify the system (19–22) and allow for an insight into the underlying physics.

First let us assume an external field given by

$$\mathbf{A}(t) = \mathbf{A}_0\sin(\omega t),\;\; \mathbf{E}(t) = -\frac{\partial}{\partial t}\mathbf{A}(t) = \mathbf{E}_0\cos(\omega t),\; \mathbf{E}_0 = -\omega\mathbf{A}_0. \qquad (26)$$

Further, we look at Eq. (21). Introducing the approximations (i) neglect of Hartree–Fock and Pauli blocking renormalizations, (ii) replacement of scattering states $|\mathbf{p}+\rangle$ by free momentum states $|\mathbf{p}\rangle$, and (iii) neglect of binary correlations in the scattering state, i.e.,

$$F_{ff}(\mathbf{P},\mathbf{p},\mathbf{p}';t)\approx F_1(\mathbf{P},\mathbf{p};t)F_2(\mathbf{P},\mathbf{p};t)(2\pi\hbar)^3\delta(\mathbf{p}-\mathbf{p}'),$$

this equation can be written as

$$\left[i\hbar\frac{\partial}{\partial t}-(E(\mathbf{p})-E_b)+\frac{e}{\mu}\mathbf{A}(t)\cdot\mathbf{p}\right]F_{fb}(\mathbf{P},\mathbf{p};t)$$
$$= -\frac{e}{\mu}\mathbf{A}(t)\cdot\mathbf{p}\,\varphi_b(\mathbf{p})\{F_{bb}(\mathbf{P};t)-F_1(\mathbf{P},\mathbf{p};t)F_2(\mathbf{P},\mathbf{p};t)\}. \qquad (27)$$

The formal solution of Eq. (27) is given by

$$F_{fb}(\mathbf{P},\mathbf{p};t) = -\frac{1}{i\hbar}\frac{e}{\mu}\int_{t_0}^{t}d\bar{t}\,e^{-\frac{i}{\hbar}(E(\mathbf{p})-E_b)(t-\bar{t})}e^{\frac{i}{\hbar}\frac{e}{\mu}\mathbf{p}\cdot\int_{\bar{t}}^{t}d\bar{\bar{t}}\mathbf{A}(\bar{\bar{t}})}\mathbf{A}(\bar{t})\cdot\mathbf{p}\,\varphi_b(\mathbf{p})$$
$$\times\{F_{bb}(\mathbf{P};\bar{t})-F_1(\mathbf{P},\mathbf{p};\bar{t})F_2(\mathbf{P},\mathbf{p};\bar{t})\}. \qquad (28)$$

For monochromatic fields (Eqs. (26)), the integral in the second exponent on the r.h.s. of (28) can be solved:

$$\frac{e}{\mu}\mathbf{p}\cdot\int_{\bar{t}}^{t} d\bar{\bar{t}}\mathbf{A}(\bar{\bar{t}}) = \frac{\mathbf{v}_0\cdot\mathbf{p}}{\omega}\left(\cos\omega t - \cos\omega\bar{t}\right), \tag{29}$$

where $\mathbf{v}_0$ is the so-called quiver velocity, $\mathbf{v}_0 = \frac{e\mathbf{E}_0}{\mu\omega}$.

Keeping in mind that the field is switched on adiabatically, i.e. at $t_0 \to -\infty$, we have to write for the vector potential

$$\mathbf{A}(t) = \lim_{\epsilon\to 0}\mathbf{A}_0\frac{1}{2i}\left[e^{i(\omega-i\epsilon)t} - e^{-i(\omega+i\epsilon)t}\right].$$

Assuming that the time dependence of the field amplitude and the distribution functions is weak compared to the very fast field oscillations, they can be taken out of the integral in (28). Then, using the relation

$$e^{\pm iz\cos\omega t} = \sum_{n=-\infty}^{\infty}(\pm i)^n J_n(z)e^{\pm in\omega t},$$

we obtain

$$\begin{aligned}F_{fb}(\mathbf{P},\mathbf{p};t) &= \frac{1}{2\hbar}\frac{e}{\mu}e^{-\frac{i}{\hbar}(E(\mathbf{p})-E_b)t}\sum_{k=-\infty}^{\infty} i^k J_k\left(\frac{\mathbf{v}_0\cdot\mathbf{p}}{\hbar\omega}\right)e^{ik\omega t}\\ &\times\sum_{n=-\infty}^{\infty}(-i)^n J_n\left(\frac{\mathbf{v}_0\cdot\mathbf{p}}{\hbar\omega}\right)\int_{t_0}^{t} d\bar{t}\, e^{\frac{i}{\hbar}(E(\mathbf{p})-E_b)\bar{t}}e^{-in\omega\bar{t}}\left[e^{i(\omega-i\epsilon)\bar{t}} - e^{-i(\omega+i\epsilon)\bar{t}}\right]\\ &\times\mathbf{A}_0\cdot\mathbf{p}\,\varphi_b(\mathbf{p})\left[F_{bb}(\mathbf{P};t) - F_1(\mathbf{P},\mathbf{p};t)F_2(\mathbf{P},\mathbf{p};t)\right].\end{aligned} \tag{30}$$

The time integration can be carried out, and after an index shift $k = n + l$ we obtain the result

$$\begin{aligned}F_{fb}(\mathbf{P},\mathbf{p};t) &= -\frac{e}{2\mu}\sum_{n=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} i^{l-1}J_n\left(\frac{\mathbf{v}_0\cdot\mathbf{p}}{\hbar\omega}\right)J_{n+l}\left(\frac{\mathbf{v}_0\cdot\mathbf{p}}{\hbar\omega}\right)\\ &\times\left[\frac{e^{i((l+1)\omega-i\epsilon)t}}{E_b - E(\mathbf{p}) + (n-1)\hbar\omega + i\hbar\epsilon} - \frac{e^{i((l-1)\omega-i\epsilon)t}}{E_b - E(\mathbf{p}) + (n+1)\hbar\omega + i\hbar\epsilon}\right]\\ &\times\mathbf{A}_0\cdot\mathbf{p}\,\varphi_b(\mathbf{p})\left[F_{bb}(\mathbf{P};t) - F_1(\mathbf{P},\mathbf{p};t)F_2(\mathbf{P},\mathbf{p};t)\right].\end{aligned} \tag{31}$$

This formula can be rewritten once more by appropriate index shifts and using the Bessel function relations

$$J_{-n}(z) = (-1)^n J_n(z)\ ; \qquad J_{n-1}(z) + J_{n+1}(z) = \frac{2n}{z}J_n(z). \tag{32}$$

We arrive at

$$\begin{aligned}F_{fb}(\mathbf{P},\mathbf{p};t) &= \frac{e}{\mu}\sum_{n=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} i^l\frac{n\hbar\omega}{\mathbf{v}_0\cdot\mathbf{p}}J_n\left(\frac{\mathbf{v}_0\cdot\mathbf{p}}{\hbar\omega}\right)J_{n+l}\left(\frac{\mathbf{v}_0\cdot\mathbf{p}}{\hbar\omega}\right)\\ &\times\frac{e^{i(l\omega-i\epsilon)t}}{E_b - E(\mathbf{p}) + n\hbar\omega + i\hbar\epsilon}\mathbf{A}_0\cdot\mathbf{p}\,\varphi_b(\mathbf{p})\left[F_{bb}(\mathbf{P};t) - F_1(\mathbf{P},\mathbf{p};t)F_2(\mathbf{P},\mathbf{p};t)\right].\end{aligned} \tag{33}$$

As can be seen from this result, the field causes interesting physical effects. First, the sum over l indicates the generation of higher harmonics of the field frequency, cf. the term $e^{i(l\omega-i\epsilon)t}$. On the other hand, the sum over n reflects the absorption or emission of multiple photons, i.e. multiphoton ionization, which finds its expression in the denominator $E_b - E(\mathbf{p}) + n\hbar\omega + i\hbar\epsilon$. Similar effects have been found, e.g., in [11, 12, 13].

4.2. Weak fields

So far, we have considered situations where the field can be of arbitrary strength[1]. However, if we want to describe situations where the field is weak, far-reaching simplifications are possible.

For that purpose, we look again at Eq. (31). The properties of the Bessel functions $J_n(z)$ show that for small arguments (i.e. for weak fields and, therefore, small quiver velocities $\mathbf{v}_0$), the main contribution to the series is given by its zeroth element. Therefore, we restrict (31) to $n = l = 0$ and obtain

$$F_{fb}(\mathbf{P},\mathbf{p};t) = -\frac{e}{2i\mu}\left[\frac{e^{i(\omega-i\epsilon)t}}{E_b - E(\mathbf{p}) - \hbar\omega + i\hbar\epsilon} - \frac{e^{-i(\omega+i\epsilon)t}}{E_b - E(\mathbf{p}) + \hbar\omega + i\hbar\epsilon}\right]$$
$$\times \mathbf{A}_0 \cdot \mathbf{p}\,\varphi_b(\mathbf{p})\left[F_{bb}(\mathbf{P};t) - F_1(\mathbf{P},\mathbf{p};t)F_2(\mathbf{P},\mathbf{p};t)\right]. \tag{34}$$

Let us consider now some properties of the transition function more in detail. We can represent (34) in the form

$$F_{fb}(\mathbf{P},\mathbf{p};t) = F_{fb}(\mathbf{P},\mathbf{p};\omega)e^{-i(\omega+i\epsilon)t} + \tilde{F}_{fb}(\mathbf{P},\mathbf{p};\omega)e^{i(\omega-i\epsilon)t}$$

Furthermore, we introduce an alternative representation of the Rabi energies. Taking into account (26) and the relation

$$\langle +\mathbf{p}|\,\mathbf{p}\,|n\rangle = \mu\,\frac{i}{\hbar}\,\langle +\mathbf{p}|\,[H,\mathbf{r}]\,|n\rangle = \mu\,\frac{i}{\hbar}(E(\mathbf{p}) - E_b)\,\langle +\mathbf{p}|\,\mathbf{r}\,|n\rangle\,,$$

we can replace

$$\begin{aligned} -\frac{e}{i\mu}\mathbf{A}_0 \cdot \mathbf{p}\,\varphi_b(\mathbf{p}) \quad &\rightarrow \quad \frac{E(\mathbf{p}) - E_b}{\hbar\omega}\mathbf{d}_{fb}(\mathbf{p}) \cdot \mathbf{E}_0, \\ \mathbf{d}_{fb}(\mathbf{p}) \quad &= \quad -e\;\langle +\mathbf{p}|\,\mathbf{r}\,|n\rangle\,. \end{aligned} \tag{35}$$

Here, $\mathbf{d}_{fb}$ is the matrix element of the electric dipole operator $\mathbf{d} = -e\mathbf{r}$.

Then, using the Dirac identity, for the imaginary part of $F_{fb}(\mathbf{P},\mathbf{p};\omega)$ follows

$$\begin{aligned} \mathrm{Im}F_{fb}(\mathbf{P},\mathbf{p};\omega) \quad &= \quad \frac{\pi}{2}\,\delta(E_b - E(\mathbf{p}) + \hbar\omega)\,\mathbf{d}_{fb} \cdot \mathbf{E}_0 \\ &\quad\; \times \{F_{bb}(\mathbf{P};t) - F_1(\mathbf{P},\mathbf{p};t)F_2(\mathbf{P},\mathbf{p};t)\}\,, \end{aligned} \tag{36}$$

and for the real part

$$\begin{aligned} \mathrm{Re}F_{fb}(\mathbf{P},\mathbf{p};\omega) \quad &= \quad \frac{1}{2}\frac{P}{E_b - E(\mathbf{p}) + \hbar\omega}\;\frac{E(\mathbf{p}) - E_b}{\hbar\omega}\;\mathbf{d}_{fb} \cdot \mathbf{E}_0 \\ &\quad\; \times \{F_{bb}(\mathbf{P};t) - F_1(\mathbf{P},\mathbf{p};t)F_2(\mathbf{P},\mathbf{p};t)\}\,. \end{aligned} \tag{37}$$

Looking at (36), a feature mentioned already above becomes more obvious. The argument of the energy conserving delta function shows that the effective ionization energy $E(\mathbf{p}) - E_b$ is brought up by the photon energy $\hbar\omega$. In the weak field case, that is just the energy of a single photon in contrast to the case of arbitrary fields where an infinite number of multiphoton contributions occurs, cf. Eq. (33).

[1] Note that our considerations are restricted to the nonrelativistic case.

5. Absorption and ionization

5.1. Electrical current and absorption coefficient

Now we are able to consider the current density and other interesting physical quantities connected to $\mathbf{j}$ like polarization, absorption and ionization coefficients. For this purpose, let us come back to the balance equation (12).

With the completeness relation for the eigenstates $|\mathbf{P}\alpha\rangle$ and the Schrödinger equation

$$1 = \sum_{\mathbf{P}\alpha} |\mathbf{P}\alpha\rangle \langle \mathbf{P}\alpha| \,, \qquad \left(H^0_{ab} - E_{\alpha\mathbf{P}}\right) |\mathbf{P}\alpha\rangle = -V_{ab} |\mathbf{P}\alpha\rangle \,,$$

we express the right hand side of the balance equation (12) for the current density in terms of the $F^{\alpha\alpha'}_{ab}(\mathbf{P};t)$. It follows

$$\frac{d\mathbf{j}(t)}{dt} - \omega_{pl}^2 \mathbf{E} = -\frac{\mathcal{V}}{2i\hbar} \sum_{ab} \int \frac{d\mathbf{p}}{(2\pi\hbar)^3} \int \frac{d\mathbf{P}}{(2\pi\hbar)^3} \sum_{\alpha\bar{\alpha}} \left(\frac{e_t \mathbf{P}}{M} + \frac{e_r \mathbf{p}}{\mu} \right) (E_{\mathbf{pP}} - E_{\alpha\mathbf{P}})$$
$$[\varphi_\alpha(\mathbf{p})\varphi^*_{\bar{\alpha}}(\mathbf{p}) F^{\alpha\bar{\alpha}}_{ab}(\mathbf{P};t) - \varphi^*_\alpha(\mathbf{p})\varphi_{\bar{\alpha}}(\mathbf{p}) F^{\bar{\alpha}\alpha}_{ab}(\mathbf{P};t)] = \sum_{ab} I_{ab}(t) \qquad (38)$$

This equation shows that the different transition processes discussed in Sec. 2 contribute to the current in a partially ionized plasma. In the following, we consider a hydrogen plasma as a simple model case. We should notice, however, that the theory can be applied to more complicated systems, too. We consider here the electron–proton part and especially the bound–free contribution ($\alpha = n = 1, \bar{\alpha} = \mathbf{p}$ and vice versa)

$$I_{ep}(t) = \frac{\mathcal{V}}{\hbar} \int \frac{d\mathbf{p}}{(2\pi\hbar)^3} \int \frac{d\mathbf{P}}{(2\pi\hbar)^3} \frac{e\mathbf{p}}{\mu} \left(E(\mathbf{p}) - E_b\right) \varphi_b(\mathbf{p}) \,\mathrm{Im}\, F_{fb}(\mathbf{P};t). \qquad (39)$$

Then we introduce the expression (33) into (39) and ignore the weak time dependence of the distribution functions and the field amplitude. Finally we integrate Eq. (38) over the time. Then follows

$$\mathbf{j}(t) = \mathbf{j}_0(t) + \sum_{l=-\infty}^{\infty} \mathbf{j}_l(\omega) e^{-il\omega t} \qquad (40)$$

which is clearly the Fourier expansion of the current in terms of all harmonics of the field frequency ω. The Fourier coefficients $\mathbf{j}_l(\omega)$ are given by

$$\begin{aligned} \mathbf{j}_l(\omega) \;=\; & -\frac{\mathcal{V}}{2} \int \frac{d\mathbf{p}}{(2\pi\hbar)^3} \int \frac{d\mathbf{P}}{(2\pi\hbar)^3} \frac{e}{\mu} \frac{E(\mathbf{p}) - E_b}{l\hbar\omega} \mathbf{p}\, \varphi_b(\mathbf{p}) \frac{e}{\mu} \mathbf{A}_0 \cdot \mathbf{p}\, \varphi_b(\mathbf{p}) \\ & \times [F_{bb}(\mathbf{P};t) - F_e(\mathbf{P},\mathbf{p};t) F_p(\mathbf{P},\mathbf{p};t)] \sum_{n=-\infty}^{\infty} i^l \frac{n\hbar\omega}{\mathbf{v}_0 \cdot \mathbf{p}} J_n\left(\frac{\mathbf{v}_0 \cdot \mathbf{p}}{\hbar\omega}\right) J_{n-l}\left(\frac{\mathbf{v}_0 \cdot \mathbf{p}}{\hbar\omega}\right) \\ & \times \left[\frac{(-1)^l}{E_b - E(\mathbf{p}) + n\hbar\omega + i\hbar\epsilon} + \frac{1}{E_b - E(\mathbf{p}) - n\hbar\omega - i\hbar\epsilon} \right] \end{aligned} \qquad (41)$$

Furthermore, $\mathbf{j}_0(t) = \omega_{pl}^2 \int_{-\infty}^{t} dt' \mathbf{E}(t')$ is the current of the collisionless plasma.

As already mentioned above, the current determines several physical quantities, e.g. the polarization given by Eq. (1). Here, we will consider the energy transfer $\mathbf{j} \cdot \mathbf{E}$ between the field and the plasma. For the field $\mathbf{E}$ we assume the time dependence (26). We consider the dissipation of the energy averaged over one cycle of oscillation. Then we obtain after a simple calculation

$$\langle \mathbf{j} \cdot \mathbf{E} \rangle = \frac{1}{T} \int_{t-T}^{t} dt' \, \mathbf{j}(t) \cdot \mathbf{E}(t) = \mathbf{E}_0 \cdot \mathrm{Re}\, \mathbf{j}_1(\omega). \qquad (42)$$

Now we introduce the absorption coefficient for the bound–free transition by

$$\alpha_{bf}(\omega) = \frac{1}{c\,\varepsilon_0} \frac{\langle \mathbf{j} \cdot \mathbf{E} \rangle}{\langle \mathbf{E}^2 \rangle} \tag{43}$$

With the help of the Dirac identity, the substitution $n \to -n$ in the second term and the Bessel function relations (32), after some algebra follows

$$\alpha_{bf}(\omega) = \frac{\mathcal{V}}{c\,\varepsilon_0 \hbar} \frac{2\pi}{E_0^2} \sum_{n=-\infty}^{\infty} (n\hbar\omega)^3 \int \frac{d\mathbf{p}}{(2\pi\hbar)^3} \int \frac{d\mathbf{P}}{(2\pi\hbar)^3} \frac{E(\mathbf{p}) - E_b}{n\hbar\omega} |\varphi_b(\mathbf{p})|^2$$
$$\times J_n^2 \left(\frac{\mathbf{v}_0 \cdot \mathbf{p}}{\hbar\omega} \right) \delta(E(\mathbf{p}) - E_b - n\hbar\omega) \left[F_{bb}(\mathbf{P}; t) - F_e(\mathbf{P}, \mathbf{p}; t) F_p(\mathbf{P}, \mathbf{p}; t) \right] \tag{44}$$

The quantity $\alpha_{bf}(\omega)$ describes the process of the absorption of n photons in the process of ionization of atoms.

Using $\frac{1}{4}(i\mathbf{v}_0 \cdot \mathbf{p}\varphi_b(\mathbf{p}))^2 = \frac{1}{4}\left(\frac{E(\mathbf{p})-E_b}{\hbar\omega}\mathbf{d}_{fb}(\mathbf{p}) \cdot \mathbf{E}_0\right)$ and introducing again the dipole matrix element, the above result gives in the linear response case

$$\alpha_{bf}(\omega) = \mathcal{V} \frac{4\pi^2\omega}{3c\hbar} \int \frac{d\mathbf{p}}{(2\pi\hbar)^3} \int \frac{d\mathbf{P}}{(2\pi\hbar)^3} |\mathbf{d}_{fb}(\mathbf{p})|^2$$
$$\times \left[F_{bb}(\mathbf{P}; t) - F_e(\mathbf{P}, \mathbf{p}; t) F_p(\mathbf{P}, \mathbf{p}; t) \right] \delta(E(\mathbf{p}) - E_b - \hbar\omega). \tag{45}$$

Here we assumed isotropy so that the tensor $\mathbf{d}_{fb}(\mathbf{p}) \otimes \mathbf{d}_{fb}(\mathbf{p})$ is diagonal with the diagonal elements $\frac{1}{3}|\mathbf{d}_{fb}|^2$.

Thus, we have shown that from our general equations, in the weak field case, there follow formulas which have the form of the well-known relations for transitions between atomic levels [8] or interband transitions in semiconductors [14]. Therefore, our approach reproduces the relations of linear response theory.

In order to determine the absorption coefficient we need the diagonal elements of the density matrix which have to be calculated from the corresponding equations (19) and (20). The assumption of thermodynamic equilibrium represents a significant simplification. The distributions are now given by

$$F_{bb}(\mathbf{P}; t) = F_{bb}(\mathbf{P}) = \exp\left\{ -\frac{1}{k_{\mathrm{B}}T} \left[E_b + \frac{\mathbf{P}^2}{2M} - \mu_A \right] \right\} \tag{46}$$
$$F_{e/p}(\mathbf{P}, \mathbf{p}; t) = F_{e/p}(\mathbf{P}, \mathbf{p}) = \exp\left\{ -\frac{1}{k_{\mathrm{B}}T} \left[\frac{1}{2m_{e/p}} \left(\frac{m_{e/p}}{M}\mathbf{P} \pm \mathbf{p} \right)^2 - \mu_{e/p} \right] \right\}, \tag{47}$$

where μ_A and $\mu_{e/p}$ are the chemical potentials of the atoms and the free particles, respectively, which are connected by $\mu_A = \mu_e + \mu_p$. Using $E(\mathbf{p}) = \frac{p^2}{2\mu}$ and the normalization condition $\int \frac{d\mathbf{P}}{(2\pi\hbar)^3} F_{bb}(\mathbf{P}) = n_A$, with n_A being the density of atoms, from Eq. (45) we obtain easily

$$\alpha_{bf}(\omega) = \mathcal{V} \frac{4\pi^2\omega}{3c\hbar} n_A \int \frac{d\mathbf{p}}{(2\pi\hbar)^3} |\mathbf{d}_{fb}(\mathbf{p})|^2 \left[1 - e^{-\frac{\hbar\omega}{k_{\mathrm{B}}T}} \right] \delta(E(\mathbf{p}) - E_b - \hbar\omega). \tag{48}$$

5.2. Rate coefficients

The absorption of photons is, under the condition $E(\mathbf{p}) - E_b = n\hbar\omega$, connected with a bound–free transition, i.e. the ionization of the atoms. The process of ionization, of course, changes the plasma composition. From the macroscopic point of view, the time dependence of the densities of the plasma particles is given by rate equations. For the change of the atomic density $n_A(t)$ we have

$$\frac{\partial}{\partial t} n_A(t) = -\alpha(t) n_A(t) + \beta(t) n_e(t) n_p(t). \tag{49}$$

Here, $n_e(t)$ and $n_p(t)$ are the densities of free particles (electrons and protons), and the coefficients $\alpha(t)$ and $\beta(t)$ are the rate coefficients.

From the microscopic point of view, the rate equation (49) follows from the equation for the occupation of the atoms F_{bb} (19),

$$\frac{\partial}{\partial t} F_{bb}(\mathbf{P};t) = \frac{2}{\hbar} \sin\omega t \int \frac{d\mathbf{p}}{(2\pi\hbar)^3} \mathbf{v}_0 \cdot \mathbf{p}\, \varphi_b(\mathbf{p})\, \mathrm{Im}\, F_{fb}(\mathbf{P},\mathbf{p};t). \tag{50}$$

Inserting the solution (33) and using the Dirac identity, we arrive at

$$\begin{aligned} \frac{\partial}{\partial t} F_{bb}(\mathbf{P};t) &= -\frac{2}{\hbar} \sin\omega t \int \frac{d\mathbf{p}}{(2\pi\hbar)^3} \mathbf{v}_0 \cdot \mathbf{p}\, |\varphi_b(\mathbf{p})|^2 \sum_{n=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} n\hbar\omega\, J_n\left(\frac{\mathbf{v}_0 \cdot \mathbf{p}}{\hbar\omega}\right) J_{n+l}\left(\frac{\mathbf{v}_0 \cdot \mathbf{p}}{\hbar\omega}\right) \\ &\times \left\{ \sin\left[l\left(\omega t + \frac{\pi}{2}\right)\right] \frac{P}{E_b - E(\mathbf{p}) + n\hbar\omega} - \pi \cos\left[l\left(\omega t + \frac{\pi}{2}\right)\right] \delta(E(\mathbf{p}) - E_b - n\hbar\omega) \right\} \\ &\times \left[F_{bb}(\mathbf{P};t) - F_e(\mathbf{P},\mathbf{p};t) F_p(\mathbf{P},\mathbf{p};t)\right]. \end{aligned} \tag{51}$$

In the following, we assume Maxwellian distributions for the free particles, i.e.

$$F_{e/p}(\mathbf{P},\mathbf{p};t) = \frac{n_{e/p}(t)\Lambda_{e/p}^3}{2} e^{-\frac{1}{2m_{e/p}k_\mathrm{B}T}\left(\frac{m_{e/p}}{M}\mathbf{P} \pm \mathbf{p}\right)^2},$$

with the thermal wavelength $\Lambda_{e/p} = \sqrt{\frac{2\pi\hbar^2}{m_{e/p}k_\mathrm{B}T}}$. Integrating Eq. (51) over $\mathbf{P}$, we obtain the rate equation (49) for the density of the atoms $n_A(t)$. Obviously, the ionization coefficient $\alpha(t)$ is then given by the microscopic expression

$$\begin{aligned} \alpha(t) &= \frac{2}{\hbar} \sin\omega t \int \frac{d\mathbf{p}}{(2\pi\hbar)^3} \mathbf{v}_0 \cdot \mathbf{p}\, |\varphi_b(\mathbf{p})|^2 \sum_{n=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} n\hbar\omega\, J_n\left(\frac{\mathbf{v}_0 \cdot \mathbf{p}}{\hbar\omega}\right) J_{n+l}\left(\frac{\mathbf{v}_0 \cdot \mathbf{p}}{\hbar\omega}\right) \\ &\times \left\{ \sin\left[l\left(\omega t + \frac{\pi}{2}\right)\right] \frac{P}{E_b - E(\mathbf{p}) + n\hbar\omega} - \pi \cos\left[l\left(\omega t + \frac{\pi}{2}\right)\right] \delta(E(\mathbf{p}) - E_b - n\hbar\omega) \right\}, \end{aligned} \tag{52}$$

and the recombination coefficient follows from

$$\begin{aligned} \beta(t) &= \frac{\Lambda_r^3}{2\hbar} \sin\omega t \int \frac{d\mathbf{p}}{(2\pi\hbar)^3} \mathbf{v}_0 \cdot \mathbf{p}\, |\varphi_b(\mathbf{p})|^2 \sum_{n=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} n\hbar\omega\, J_n\left(\frac{\mathbf{v}_0 \cdot \mathbf{p}}{\hbar\omega}\right) J_{n+l}\left(\frac{\mathbf{v}_0 \cdot \mathbf{p}}{\hbar\omega}\right) \\ &\times \left\{ \sin\left[l\left(\omega t + \frac{\pi}{2}\right)\right] \frac{P}{E_b - E(\mathbf{p}) + n\hbar\omega} - \pi \cos\left[l\left(\omega t + \frac{\pi}{2}\right)\right] \delta(E(\mathbf{p}) - E_b - n\hbar\omega) \right\} \\ &\times e^{-\frac{p^2}{2\mu k_\mathrm{B}T}} \end{aligned} \tag{53}$$

with $\Lambda_r = \sqrt{\frac{2\pi\hbar^2}{\mu k_\mathrm{B}T}}$. It is useful to average again over one period of the field oscillation. Using the relations $\frac{1}{T}\int_0^T dt\, \sin\omega t \cos l\omega t = 0$, $\frac{1}{T}\int_0^T dt\, \sin\omega t \sin l\omega t = \frac{l}{2}\delta_{|l|1}$, we get the averaged coefficients

$$\bar{\alpha} = \frac{2\pi}{\hbar} \int \frac{d\mathbf{p}}{(2\pi\hbar)^3} |\varphi_b(\mathbf{p})|^2 \sum_{n=-\infty}^{\infty} (n\hbar\omega)^2 J_n^2\left(\frac{\mathbf{v}_0 \cdot \mathbf{p}}{\hbar\omega}\right) \delta(E(\mathbf{p}) - E_b - n\hbar\omega), \tag{54}$$

$$\bar{\beta} = \frac{\pi\Lambda_r^3}{2\hbar} \int \frac{d\mathbf{p}}{(2\pi\hbar)^3} |\varphi_b(\mathbf{p})|^2 \sum_{n=-\infty}^{\infty} (n\hbar\omega)^2 J_n^2\left(\frac{\mathbf{v}_0 \cdot \mathbf{p}}{\hbar\omega}\right) \delta(E(\mathbf{p}) - E_b - n\hbar\omega) e^{-\frac{p^2}{2\mu k_\mathrm{B}T}}. \tag{55}$$

This strongly simplified expression for the ionization coefficient is essentially equivalent to results from atomic physics [2, 3, 5]. However, in contrast to the results of Faisal and Reiss, no effect of the A^2 term occurs.

6. Conclusions and outlook

In this paper we have presented an approach to describe the process of ionization and recombination in partially ionized plasmas under the influence of an external electromagnetic field. The resulting system of equations represents a generalization of the Bloch equations known from atomic and semiconductor physics. The central quantity is the transition function describing transitions between bound and scattering states. We have discussed this function in several physical situations. By means of the transition function we have investigated the energy transfer between field and plasma and the field-driven change of the plasma composition. In the case of strong fields, interesting physical effects like multiphoton ionization and higher harmonics generation occur.

A challenging task for future research is, of course, the selfconsistent numerical solution of the full system of plasma Bloch equations, Eqs. (19–22). Then all quantities of interest—distributions of atoms and electron–ion pairs as well as transition functions—can be analyzed in their temporal evolution even for ultrashort times immediately after the laser switch-on.

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