

Relativistic Kadanoff-Baym equations for correlated initial states and baryogenesis

Mathias Garny (TU München)

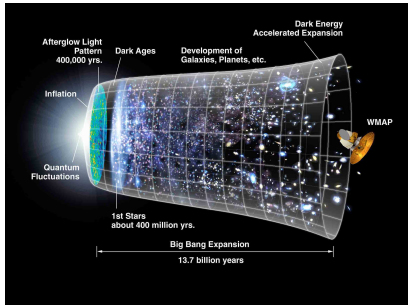
KBE2010, CAU Kiel, 23–26.03.2010

based on Phys.Rev.D80:085011,2009 with Markus Michael Müller
ongoing work with Urko Reinos

Relativistic Kadanoff-Baym equations for correlated initial states and baryogenesis

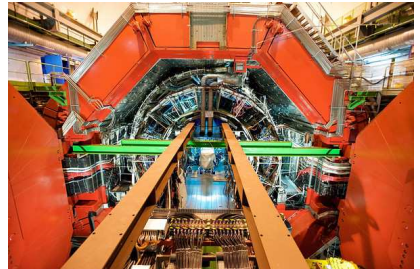
- Motivation: Baryogenesis
- Relativistic Kadanoff-Baym Equations
- UV divergences, renormalization and Non-Gaussian ICs
- Results for $\lambda\Phi^4$ 2PI 3-loop
- Numerical methods

Nonequilibrium dynamics at high energy



Heavy Ion Collisions

- LHC: ALICE
- RHIC



Early universe

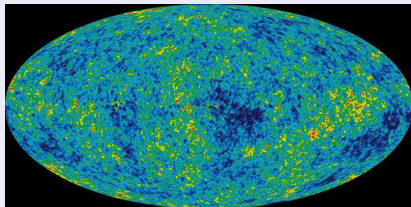
- Reheating after Inflation
- Baryogenesis
- ...

Baryon asymmetry of the universe

- Our galaxy consists of matter: $\bar{p}/p \lesssim 10^{-3}$
- No annihilations observed

Asymmetry parameter

$$\eta = \frac{n_b - n_{\bar{b}}}{s}$$



n_b = baryon density

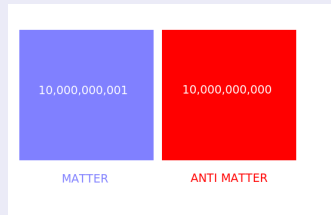
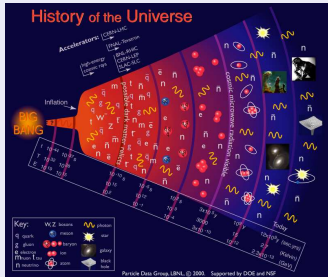
$n_{\bar{b}}$ = anti-baryon density

s = entropy density

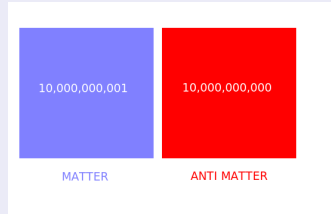
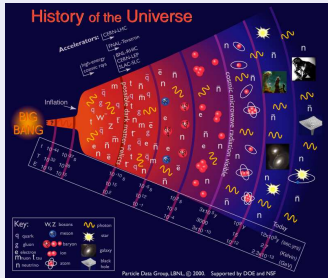
$$4.7 \cdot 10^{-10} < \eta < 6.5 \cdot 10^{-10} \text{ (95\% CL)}$$

Baryogenesis

Baryon asymmetry of the universe



Baryon asymmetry of the universe

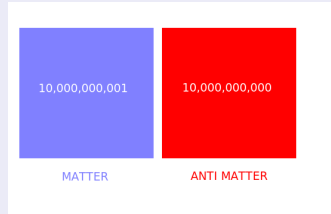
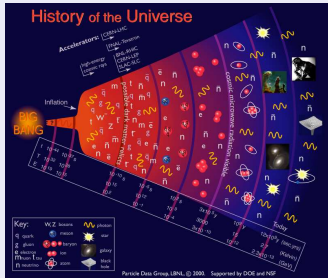


(1) Initial baryon asymmetry after Big Bang

Problem:

- Diluted by inflation
- Washed out by $\Delta B \neq 0$ processes at high energy

Baryon asymmetry of the universe



(1) Initial baryon asymmetry after Big Bang

Problem:

- Diluted by inflation
- Washed out by $\Delta B \neq 0$ processes at high energy

(2) Dynamical creation: **Baryogenesis**

Baryogenesis: three Sakharov conditions

Sakharov 1967

- baryon number violation: $\langle B \rangle \neq \text{const.}$
- CP violation: $\Gamma(i \rightarrow f) \neq \Gamma(\bar{i} \rightarrow \bar{f})$
- **deviation from thermal equilibrium**: $\Gamma(i \rightarrow f) \neq \Gamma(f \rightarrow i)$

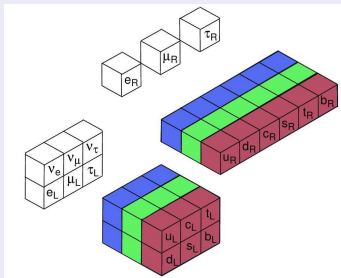
Baryogenesis

Baryogenesis: three Sakharov conditions

Sakharov 1967

- baryon number violation: $\langle B \rangle \neq \text{const.}$
- CP violation: $\Gamma(i \rightarrow f) \neq \Gamma(\bar{i} \rightarrow \bar{f})$
- **deviation from thermal equilibrium**: $\Gamma(i \rightarrow f) \neq \Gamma(f \rightarrow i)$

Baryogenesis within the Standard Model of Particles ?



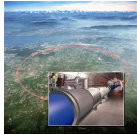
- B-violation for $T > T_{EW}$

$$\Delta B = \Delta L$$

- CP-violation in quark mixing
→ $K^0/\bar{K}^0$ decay
- Expanding universe

But: much too weak...

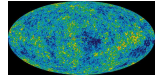
Baryogenesis



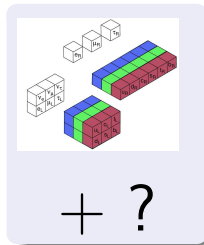
Collider exp.



Baryogenesis



$$4.7 \cdot 10^{-10} < \eta < 6.5 \cdot 10^{-10}$$

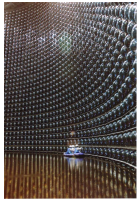


Dark matter

$$\frac{\rho_{\text{dark matter}}}{\rho_0} \simeq 0.23$$



Neutrino
exp.



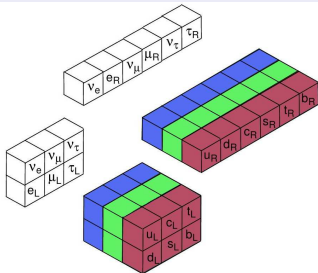
Superkamiokande
Neutrino Detector

Baryogenesis

Baryogenesis: three Sakharov conditions

- baryon number violation: $\langle B \rangle \neq \text{const.}$
- CP violation: $\Gamma(i \rightarrow f) \neq \Gamma(\bar{i} \rightarrow \bar{f})$
- **deviation from thermal equilibrium**: $\Gamma(i \rightarrow f) \neq \Gamma(f \rightarrow i)$

Baryogenesis in the SM + Right-handed neutrinos $(\nu_R)_{e,\mu,\tau}$



- B- via L-violation $M_R \bar{\nu}_R \nu_R^c \rightarrow 0\nu\beta\beta: (A, Z) \rightarrow (A, Z + 2) + 2e^-$ (Gerda, Nemo, Exo, ...)
- CP-violation in ν -mixing $\rightarrow \nu$ -oscillation (Double Chooz, Daya Bay, ...)
- Expanding universe

Leptogenesis

Out-of-equilibrium decay of heavy right-handed neutrino ν_R

$$\mathcal{M}_{\nu_R, i \rightarrow \ell_{L, \alpha} h^\dagger} = \text{---} \overset{y_{i\alpha}}{\text{---}} + \dots$$

$$\mathcal{M}_{\nu_R, i \rightarrow \ell_{L, \alpha}^c h} = \text{---} \overset{y_{i\alpha}^*}{\text{---}} + \dots$$

Leptogenesis

Out-of-equilibrium decay of heavy right-handed neutrino ν_R

$$\mathcal{M}_{\nu_R, i \rightarrow \ell_{L, \alpha} h^\dagger} = \text{tree} + \text{loop} + \dots$$

$$\mathcal{M}_{\nu_R, i \rightarrow \ell_{L, \alpha}^c h} = \text{tree} + \text{loop} + \dots$$

CP violation in decay described by **loop process**

$$\Gamma(\nu_{R,i} \rightarrow \ell_{L,\alpha} h^\dagger) - \Gamma(\nu_{R,i} \rightarrow \ell_{L,\alpha}^c h) \sim \text{Im}(y_{i\alpha} y_{i\beta} y_{j\alpha}^* y_{j\beta}^*) \cdot \text{Im} \left(\text{loop diagram} \right)$$

Baryogenesis via Leptogenesis

- CP violation in decay described by **loop process**
- **deviation from thermal equilibrium**

Quantum nonequilibrium effects ?

The semi-classical approach

Boltzmann equation for leptogenesis

$$\begin{aligned} p^\alpha \mathcal{D}_\alpha f_{\ell_L}(t, \mathbf{x}, \mathbf{p}) &= \int d\Pi_{\nu_R} d\Pi_h \\ &\times (2\pi)^4 \delta(p_{\ell_L} + p_h - p_{\nu_R}) \\ &\times \left[|\mathcal{M}|_{\nu_R \rightarrow \ell_L h^\dagger}^2 f_{\nu_R} (1 - f_{\ell_L}) (1 + f_h) \right. \\ &\quad \left. - |\mathcal{M}|_{\ell_L h^\dagger \rightarrow \nu_R}^2 f_{\ell_L} f_h (1 - f_{\nu_R}) \right] \end{aligned}$$



$|\mathcal{M}|^2$: microscopic interactions, **off-shell** processes

$f(t, \mathbf{x}, \mathbf{p})$: macroscopic propagation of **on-shell** particles

$$\Delta x_{\text{interaction}} \ll \lambda_{\text{mfp}}, \quad \lambda_{\text{de-Broglie}} \ll \lambda_{\text{mfp}}$$

$$1/M \ll 1/\Gamma, \quad 1/T \ll 1/(y^2 T)$$

Corrections within Boltzmann picture

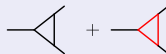
Bose-enhancement, Pauli-Blocking; kinetic (non-)equilibrium

- quantum statistical factors $1 \pm f_k$
- non-integrated Boltzmann equations

Hannestad, Basbøll 06; Garayoa, Pastor, Pinto, Rius, Vives 09; Hahn-Woernle, Plümacher, Wong 09

Medium corrections

- medium correction to decay rates



$$\epsilon = \frac{\Gamma(\nu_R \rightarrow \ell h^\dagger) - \Gamma(\nu_R \rightarrow \ell^c h)}{\Gamma(\nu_R \rightarrow \ell h^\dagger) + \Gamma(\nu_R \rightarrow \ell^c h)} = \epsilon^{\text{vac}} + \delta\epsilon^{\text{th}}(T, \dots)$$

- thermal masses

MG, Hohenegger, Kartavtsev, Lindner 09; Kiessig, Plümacher 09; Giudice, Notari, Raidal, Riotto, Stumia 04; Covi, Rius, Roulet, Vissani 98; . . .

Flavour effects

Nardi, Nir, Roulet, Racker 06; Adaba, Davidson, Ibarra, Josse-Micheaux, Losada, Riotto 06; Blanchet, diBari 06. . .

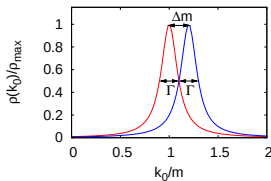
Limitations of the Boltzmann approach

- Unstable particles lead to double counting problems
[real intermediate state subtraction]



- Resonant Leptogenesis: $\Gamma \sim \Delta M$

Pilaftsis, Underwood, ...



- Quantum interference out of equilibrium

$$\epsilon = \frac{\Gamma(\nu_R \rightarrow \ell h^\dagger) - \Gamma(\nu_R \rightarrow \ell^c h)}{\Gamma(\nu_R \rightarrow \ell h^\dagger) + \Gamma(\nu_R \rightarrow \ell^c h)} \sim \text{diagram} \times \left(\text{diagram} \right)^* \sim 10^{-7}$$

Going beyond the Boltzmann picture

Statistical propagator $G_F^{ij}(x, y) = \langle \Phi^i(x) \Phi^j(y) + \Phi^j(y) \Phi^i(x) \rangle / 2$

Spectral function $G_\rho^{ij}(x, y) = i \langle \Phi^i(x) \Phi^j(y) - \Phi^j(y) \Phi^i(x) \rangle$

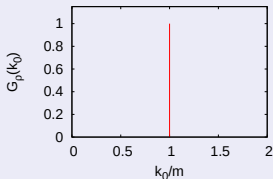
Going beyond the Boltzmann picture

Statistical propagator $G_F^{ij}(x, y) = \langle \Phi^i(x) \Phi^j(y) + \Phi^j(y) \Phi^i(x) \rangle / 2$

Spectral function $G_\rho^{ij}(x, y) = i \langle \Phi^i(x) \Phi^j(y) - \Phi^j(y) \Phi^i(x) \rangle$

Boltzmann limit

- on-shell quasi-stable particles



$$G_\rho^{ij}(k) \sim \delta^{ij} \delta(k^2 - m_i^2)$$

- equilibrium-like fluctuation-dissipation relation

$$G_F^{ij}(t, k) = \left(f_k^i(t) + \frac{1}{2} \right) G_\rho^{ij}(k)$$

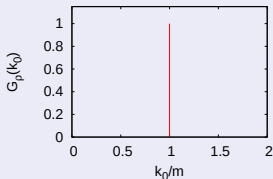
Going beyond the Boltzmann picture

Statistical propagator $G_F^{ij}(x, y) = \langle \Phi^i(x) \Phi^j(y) + \Phi^j(y) \Phi^i(x) \rangle / 2$

Spectral function $G_\rho^{ij}(x, y) = i \langle \Phi^i(x) \Phi^j(y) - \Phi^j(y) \Phi^i(x) \rangle$

Boltzmann limit

- on-shell quasi-stable particles



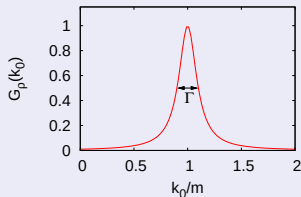
$$G_\rho^{ij}(k) \sim \delta^{ij} \delta(k^2 - m_i^2)$$

- equilibrium-like fluctuation-dissipation relation

$$G_F^{ij}(t, k) = \left(f_k^i(t) + \frac{1}{2} \right) G_\rho^{ij}(k)$$

Propagation beyond Boltzmann

- spectrum with (thermal) width



$$G_\rho^{ij}(t, k) \sim \frac{\delta^{ij} 2k_0 \Gamma_i(t)}{(k^2 - m_{th,i}^2(t))^2 + k_0^2 \Gamma_i(t)^2} + \dots$$

- on-/off-shell, cross-correlations

$$G_F^{ij}(t, k) = \begin{pmatrix} G_F^{11} & G_F^{12} \\ G_F^{21} & G_F^{22} \end{pmatrix}$$

Relativistic Kadanoff-Baym equations

Relativistic Kadanoff-Baym equations

$$\begin{aligned} (\partial_{x^0}^2 - \nabla_x^2 + m_i^2(x)) G_F^{ij}(x, y) &= \int_0^{y^0} d^4 z \Pi_F^{ik}(x, z) G_\rho^{kj}(z, y) \\ &\quad - \int_0^{x^0} d^4 z \Pi_\rho^{ik}(x, z) G_F^{kj}(z, y) \end{aligned}$$

$$(\partial_{x^0}^2 - \nabla_x^2 + m_i^2(x)) G_\rho^{ij}(x, y) = \int_{x_0}^{y^0} d^4 z \Pi_\rho^{ik}(x, z) G_\rho^{kj}(z, y)$$

$$\text{Statistical propagator} \quad G_F^{ij}(x, y) = \langle \Phi^i(x) \Phi^j(y) + \Phi^j(y) \Phi^i(x) \rangle / 2$$

$$\text{Spectral function} \quad G_\rho^{ij}(x, y) = i \langle \Phi^i(x) \Phi^j(y) - \Phi^j(y) \Phi^i(x) \rangle$$

Relativistic Kadanoff-Baym equations

Obtained from stationarity condition of the 2PI effective action

Cornwall, Jackiw, Tomboulis (1974)

$$\frac{\delta \Gamma[\phi, G]}{\delta G} = 0 \quad \Leftrightarrow \quad G^{-1} = G_0^{-1} - \Pi[G]$$

where $G(x, y) = G_F(x, y) - i/2 \operatorname{sign}_{\mathcal{C}}(x^0 - y^0) G_\rho(x, y)$

Controlled approximation...

... by truncation of the 2PI functional $\Gamma_2[\phi, G]$

Relativistic Kadanoff-Baym equations

Example: Three-loop truncation in $\lambda\Phi^4$ -theory (for $\langle\Phi\rangle = 0$)

$$\Gamma_2[G] = \text{diagram 1} + \text{diagram 2}$$

Diagram 1: A self-energy loop on a two-point vertex, represented as two circles joined at a single point.

Diagram 2: A tadpole diagram on a two-point vertex, represented as a circle with two external lines attached to the same vertex.

$$\Pi[G] = \frac{2i\delta\Gamma_2}{\delta G} = \text{diagram 3} + \text{diagram 4}$$

Diagram 3: A tadpole diagram on a three-point vertex, represented as a circle with one external line and one internal loop line attached to the same vertex.

Diagram 4: A self-energy loop on a three-point vertex, represented as a circle with two external lines and one internal loop line attached to the same vertex.

$$S = \int d^4x \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4 \right)$$

$$\text{four-point vertex} = -i\lambda\delta(x_1 - x_2)\delta(x_1 - x_3)\delta(x_1 - x_3)$$

Diagram: A four-point vertex represented by four lines meeting at a central point.

$$\text{propagator} = G(x, y)$$

Diagram: A propagator represented by a horizontal line with dots at both ends.

Relativistic Kadanoff-Baym equations

Setting-sun approximation for $\lambda\Phi^4$ -theory

$$\begin{aligned}
 \left(\square_x + m^2 + \text{red loop} \right) G_F(x, y) &= \int_0^{y^0} d^4 z \left(\text{red circle} + \text{green circle} \right) G_\rho(z, y) \\
 &\quad - \int_0^{x^0} d^4 z \left(\text{red circle} + \text{green circle} \right) G_F(z, y) \\
 \left(\square_x + m^2 + \text{red loop} \right) G_\rho(x, y) &= \int_{x_0}^{y^0} d^4 z \left(\text{red circle} + \text{green circle} \right) G_\rho(z, y)
 \end{aligned}$$

$$\begin{aligned}
 \text{red loop} &= \frac{\lambda}{2} G_F(x, x) & \text{red circle} &= -\frac{\lambda^2}{6} G_F(x, z)^3 & \text{green circle} &= -\frac{\lambda^2}{6} G_\rho(x, z)^3 \\
 \text{green circle} &= -\frac{\lambda^2}{6} G_F(x, z) G_\rho(x, z)^2 & \text{red circle} &= -\frac{\lambda^2}{6} G_F(x, z)^2 G_\rho(x, z)
 \end{aligned}$$

Relativistic Kadanoff-Baym equations

Homogeneous system

$$G(x, y) = G(x^0, y^0, \mathbf{x} - \mathbf{y}) \rightarrow G(x^0, y^0, \mathbf{k}), \quad \mathbf{k} = (k_x, k_y, k_z)$$

$$\left(\partial_{x^0}^2 + \mathbf{k}^2 + m^2 + \text{red loop} \right) G_F(x^0, y^0, \mathbf{k}) =$$

$$\int_0^{y^0} dz^0 \left(\text{red circle} + \text{green circle} \right) G_\rho(z^0, y^0, \mathbf{k})$$

$$- \int_0^{x^0} dz^0 \left(\text{red circle} + \text{green circle} \right) G_F(z^0, y^0, \mathbf{k})$$

$$\text{red loop} = \frac{\lambda}{2} \int \frac{d^3 p}{(2\pi)^3} G_F(x^0, x^0, \mathbf{p})$$

$$\text{red circle} = -\frac{\lambda^2}{6} \int \frac{d^3 p}{(2\pi)^3} \int \frac{d^3 q}{(2\pi)^3} G_F(x^0, z^0, \mathbf{p}) G_F(x^0, z^0, \mathbf{q}) G_F(x^0, z^0, \mathbf{k} - \mathbf{p} - \mathbf{q})$$

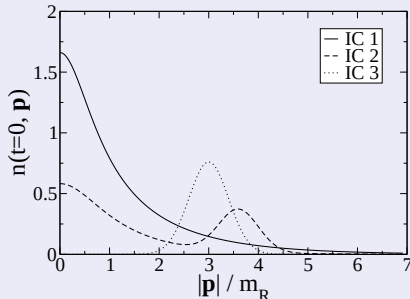
$$= -\frac{\lambda^2}{6} \int d^3 x e^{i\mathbf{k}\mathbf{x}} \left[\int \frac{d^3 p}{(2\pi)^3} e^{-i\mathbf{p}\mathbf{x}} G_F(x^0, z^0, \mathbf{p}) \right]^3 \quad \text{Danielewicz, Köhler, ...}$$

Relativistic Kadanoff-Baym equations

Initial conditions (Gaussian initial state)

Example: $\phi = \dot{\phi} = 0$,

$$\begin{aligned} G(x^0, y^0, \mathbf{p})|_{x^0=y^0=t_{init}} &= \frac{n_{\mathbf{p}}(t_{init}) + 1/2}{\omega_{\mathbf{p}}(t_{init})} \\ (\partial_{x^0} + \partial_{y^0}) G(x^0, y^0, \mathbf{p})|_{x^0=y^0=t_{init}} &= 0 \\ \partial_{x^0} \partial_{y^0} G(x^0, y^0, \mathbf{p})|_{x^0=y^0=t_{init}} &= \omega_{\mathbf{p}}(t_{init}) (n_{\mathbf{p}}(t_{init}) + 1/2) \end{aligned}$$



$$\omega_{\mathbf{p}}(t_{init}) = \sqrt{m_R^2 + \mathbf{k}^2}$$

NR nuclear coll: *Danielewicz (1983); Köhler (1994,...); ...*

- Thermalization in relativistic scalar QFT

Berges, Cox (2001); Berges (2002); Aarts, Berges (2002); Aarts, Resco (2003); Juchem, Cassing, Greiner (2004); Arrizabalaga, Smit, Tranberg (2005); Lindner, Müller (2006); Gasenzer, Pawłowski (2008); ...

- Thermalization in relativistic fermionic QFT

Berges, Borsanyi, Serreau (2003); Lindner, Müller (2008)

- Prethermalization

Berges, Borsanyi, Wetterich (2004)

- Nonequilibrium Instabilities, Parametric Resonance

Berges, Serreau (2003), Aarts, Tranberg (2007); Berges, Rothkopf, Schmidt (2008); Berges, Pruschke, Rothkopf (2009)

- 2PI renormalization *Borsanyi,Reinosa (2008); MG, Müller (2009)*

Leptogenesis/Baryogenesis [no two-time KBE analysis yet]

Buchmüller, Fredenhagen (2000); DeSimone, Riotto (2007); Anisimov, Buchmüller, Drewes, Mendizabal (2008,10); MG, Hohenegger, Kartavtsev, Lindner (2009,10); Gagnon (2009)

NR nuclear coll: Danielewicz (1983); Köhler (1994,...); ...

- Thermalization in relativistic scalar QFT

Berges, Cox (2001); Berges (2002); Aarts, Berges (2002); Aarts, Resco (2003); Juchem, Cassing, Greiner (2004); Arrizabalaga, Smit, Tranberg (2005); Lindner, Müller (2006); Gasenzer, Pawłowski (2008); ...

- Thermalization in relativistic fermionic QFT

Berges, Borsanyi, Serreau (2003); Lindner, Müller (2008)

- Prethermalization

Berges, Borsanyi, Wetterich (2004)

- Nonequilibrium Instabilities, Parametric Resonance

Berges, Serreau (2003), Aarts, Tranberg (2007); Berges, Rothkopf, Schmidt (2008); Berges, Pruscke, Rothkopf (2009)

- **2PI renormalization** *Borsanyi,Reinosa (2008); MG, Müller (2009)*

Leptogenesis/Baryogenesis [no two-time KBE analysis yet]

Buchmüller, Fredenhagen (2000); DeSimone, Riotto (2007); Anisimov, Buchmüller, Drewes, Mendizabal (2008,10); MG, Hohenegger, Kartavtsev, Lindner (2009,10); Gagnon (2009)

UV Divergences

$$\begin{aligned} \text{---} \text{---} \text{---} \text{---} \text{---} &= \frac{\lambda}{2} \int \frac{d^3 p}{(2\pi)^3} G_{F,0}(x^0, x^0, p) \\ &= \frac{\lambda}{2} \int \frac{d^3 p}{(2\pi)^3} \frac{n_p + \frac{1}{2}}{\sqrt{m^2 + \mathbf{p}^2}} \sim \Lambda^2 \end{aligned}$$

$$\int dz^0 \text{---} \text{---} \text{---} \text{---} \text{---} G(z^0, y^0, k) \sim \Lambda^2$$

UV cut-off $|\mathbf{p}| \leq k_{\max} \equiv \Lambda \quad \Rightarrow \quad \text{Quadratic divergence}$

Renormalization ?

$$\begin{aligned}
 & \left(\partial_{x^0}^2 + \mathbf{k}^2 + m_R^2 + \text{---}\bigotimes\text{---} + \text{---}\bigcirc\text{---} + \text{---}\bigotimes\text{---} \right) G_F(x^0, y^0, \mathbf{k}) = \\
 & \qquad \qquad \qquad \int_0^{y^0} dz^0 \left(\text{---}\bigcirc\text{---} + \text{---}\bigcirc\text{---} \right) G_\rho(z^0, y^0, \mathbf{k}) \\
 & \qquad \qquad \qquad - \int_0^{x^0} dz^0 \left(\text{---}\bigcirc\text{---} + \text{---}\bigcirc\text{---} \right) G_F(z^0, y^0, \mathbf{k}) \\
 & \qquad \qquad \qquad \text{---}\bigotimes\text{---} = \delta m^2 + \delta Z \mathbf{k}^2 \\
 & \qquad \qquad \qquad \text{---}\bigotimes\text{---} = \delta \lambda
 \end{aligned}$$

Nonperturbative 2PI counterterms: Hees, Knoll (2001, 2002); Blaizot, Iancu, Reinosa (2003); Berges, Borsanyi, Reinosa, Serreau (2004, 2005); Reinosa, Serreau (2006, 2007, 2009)

Problem

- Gaussian initial states
- 'bare' particles in the initial state
- Incompatible with renormalization

Why does the Gaussian initial state lead to singularities ?

$$E_{total} = E_{kin}(t) + E_{corr}(t)$$

$$E_{kin}(t) = \frac{1}{2} \int \frac{d^3 k}{(2\pi)^3} \left[\partial_{x^0} \partial_{y^0} + \mathbf{k}^2 + m_R^2 + \text{---} \otimes \text{---} \right. \\ \left. + \text{---} \bullet \text{---} + \text{---} \otimes \text{---} \right] \text{---} \bullet \big|_{x^0=y^0=t} + \delta\Lambda$$

$\delta m^2, \delta Z$

$\delta\lambda$

$$E_{corr}(t) = \int_0^t dz^0 \int \frac{d^3 k}{(2\pi)^3} \text{---} \bullet \text{---} \big|_{x^0=y^0=t}$$

- $E_{corr}(t)$ contains divergences
- $E_{kin}(t)$ contains 2PI counterterms

Berges, Borsanyi, Reinosa, Serreau (2004, 2005)

Why does the Gaussian initial state lead to singularities ?

$$E_{total} = E_{kin}(t) + E_{corr}(t)$$

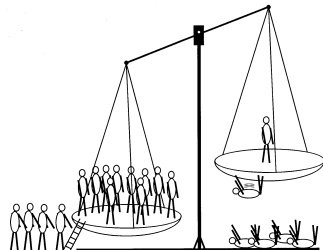
$$E_{kin}(t) = \frac{1}{2} \int \frac{d^3 k}{(2\pi)^3} \left[\partial_{x^0} \partial_{y^0} + \mathbf{k}^2 + m_R^2 + \text{---} \bigotimes \text{---} \right. \\ \left. + \text{---} \bigcirc \text{---} + \text{---} \bigotimes \text{---} \right] \text{---} \bullet \big|_{x^0=y^0=t} + \delta\Lambda$$

$\delta m^2, \delta Z$

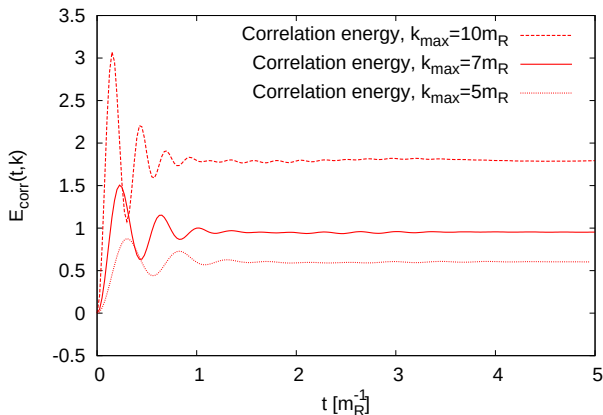
$\delta\lambda$

$$E_{corr}(t) = \int_0^t dz^0 \int \frac{d^3 k}{(2\pi)^3} \text{---} \bigcirc \text{---} \bullet \big|_{x^0=y^0=t}$$

- $E_{corr}(t)$ contains divergences
- $E_{kin}(t)$ contains 2PI counterterms
Berges, Borsanyi, Reinosa, Serreau (2004, 2005)
- $E_{corr}(t)|_{t=0} = 0$ for Gaussian initial state
- $\Rightarrow$ unbalanced divergence at $t = 0$



Why does the Gaussian initial state lead to singularities ?



Non-Gaussian initial state

- *Explicit:* non-Gaussian density matrix ρ at $t = t_{init}$
 - imaginary time-stepping $\rho = \exp(-\mathcal{O})$
 - initial n -point correlation functions

$$\langle \varphi_+ | \rho | \varphi_- \rangle = \exp \left(i \sum_{n=0}^{\infty} \int_{\mathbf{x}_i} \alpha_n^{\epsilon_1 \dots \epsilon_n}(\mathbf{x}_1, \dots, \mathbf{x}_n) \varphi_{\epsilon_1}(\mathbf{x}_1) \cdots \varphi_{\epsilon_n}(\mathbf{x}_n) \right)$$

$$\text{where } \Phi(t_{init}, \mathbf{x}) | \varphi_{\pm} \rangle = \varphi_{\pm}(\mathbf{x}) | \varphi_{\pm} \rangle \quad [\text{Calzetta, Hu}]$$

e.g. Hall, Kukharev, Tikhodeev, Danielewicz, Wagner, Schlages, Bornath, Semkat, Kremp, Bonitz, Köhler, Morozov, Röpke,...

- *Implicit*
 - external two-point source $K(x, y)$ for $t < t_{init}$
e.g. Borsanyi, Reinosa,...

Gaussian initial state

All n -point correlation functions vanish at $t = t_{init}$ for $n \geq 3$

$$\alpha_n(x_1, \dots, x_n) = 0 \quad \text{for } n \geq 3$$

Renormalized initial state

Relevant n -point correlation functions asymptotically agree with vacuum correlations at short distances [for $n \leq 4$]

$$\alpha_n(x_1, \dots, x_n) = \alpha_n^{vac/th}(x_1, \dots, x_n) + \Delta\alpha_n(x_1, \dots, x_n)$$

Initial n -point correlation functions:

Local standard vertices:

$$\begin{array}{c} \diagup \\ \bullet \\ \diagdown \end{array} = -i\lambda_R, \quad \begin{array}{c} \diagup \\ \otimes \\ \diagdown \end{array} = -i\delta\lambda$$

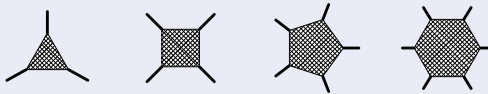
UV divergences, renormalization and non-Gaussian ICs

Initial n -point correlation functions:

Local standard vertices:

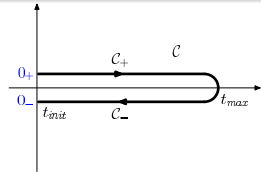
$$\begin{array}{c} \diagup \quad \diagdown \\ \bullet \end{array} = -i\lambda_R, \quad \begin{array}{c} \diagup \quad \diagdown \\ \otimes \end{array} = -i\delta\lambda$$

Effective non-local vertices: $\alpha_n(\mathbf{x}_1, \dots, \mathbf{x}_n)$



... encode the non-Gaussian initial correlations

$$\begin{aligned} \alpha_n(\mathbf{x}_1, \dots, \mathbf{x}_n) &= \sum_{\epsilon_i \in \pm} \alpha_n^{\epsilon_1, \dots, \epsilon_n}(\mathbf{x}_1, \dots, \mathbf{x}_n) \\ &\quad \times \delta(x_1^0 - 0_{\epsilon_1}) \cdots \delta(x_n^0 - 0_{\epsilon_n}) \end{aligned}$$



$$\alpha_n^{\epsilon_1, \dots, \epsilon_n}(\mathbf{x}_1, \dots, \mathbf{x}_n), \quad \epsilon_i \in \{+, -\} \quad \Leftrightarrow \quad \text{IC for } (\partial_{x_1^0})^{\epsilon_1} \cdots (\partial_{x_n^0})^{\epsilon_n} V_n(x_1, \dots, x_n), \quad c_i \in \{0, 1\}$$

UV divergences, renormalization and non-Gaussian ICs

Example: Initial 4-point correlation, 2PI three-loop truncation

$$\begin{aligned}\Gamma_2[G, \alpha_4] &= \text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \text{diagram 4} + \text{diagram 5} \\ \Pi[G, \alpha_4] &= \underbrace{\text{diagram 6} + \text{diagram 7}}_{\Pi_{Gauss}} + \underbrace{\text{diagram 8} + \text{diagram 9}}_{\Pi_{\lambda\alpha}} + \underbrace{\text{diagram 10}}_{\Pi_{\alpha\lambda}} + \underbrace{\text{diagram 11} + \text{diagram 12} + \text{diagram 13}}_{\Pi_{\alpha\alpha}}\end{aligned}$$

The diagrams represent Feynman diagrams for the two-point function Γ_2 and the self-energy Π at the 2PI three-loop truncation level. The diagrams for Γ_2 are: a tadpole, a tadpole with a shaded square, a bubble, a bubble with a shaded square, and a bubble with two shaded squares. The diagrams for Π are: a tadpole, a bubble, a bubble with a shaded square, a bubble with a shaded square, a tadpole with a shaded square, and a bubble with two shaded squares. The brackets below the Π diagrams are labeled Π_{Gauss} , $\Pi_{\lambda\alpha}$, $\Pi_{\alpha\lambda}$, and $\Pi_{\alpha\alpha}$ respectively.

UV divergences, renormalization and non-Gaussian ICs

Example: Initial 4-point correlation, 2PI three-loop truncation

$$\begin{aligned}
 \Gamma_2[G, \alpha_4] &= \text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \text{diagram 4} + \text{diagram 5} \\
 \Pi[G, \alpha_4] &= \underbrace{\text{diagram 6} + \text{diagram 7}}_{\Pi_{Gauss}} + \underbrace{\text{diagram 8} + \text{diagram 9}}_{\Pi_{\lambda\alpha}} + \underbrace{\text{diagram 10} + \text{diagram 11}}_{\Pi_{\alpha\lambda}} + \underbrace{\text{diagram 12} + \text{diagram 13}}_{\Pi_{\alpha\alpha}}
 \end{aligned}$$

The diagrams represent various Feynman-like diagrams for the 2PI three-loop truncation. The first row shows the two-point function Γ_2 with five diagrams. The second row shows the self-energy Π with six diagrams, grouped into four categories: Π_{Gauss} (Gaussian), $\Pi_{\lambda\alpha}$, $\Pi_{\alpha\lambda}$, and $\Pi_{\alpha\alpha}$ (non-Gaussian).

Initial-time surface contribution

e.g. Danielewicz, Semkat, Kremp, Bonitz, ...

For the example:

$$\Pi_{\lambda\alpha}(x, z) = -\frac{\lambda}{6} \int_{\mathcal{C}} d^4 y_{123} G(x, y_1) G(x, y_2) G(x, y_3) \alpha_4(y_1, y_2, y_3, z)$$

In general:

$$\Pi_{\lambda\alpha}(x, z) = \Pi_{\lambda\alpha}^+(x, z) \delta(z^0 - 0_+) + \Pi_{\lambda\alpha}^-(x, z) \delta(z^0 - 0_-)$$

UV divergences, renormalization and non-Gaussian ICs

$$\left(\partial_{x^0}^2 + \mathbf{k}^2 + m_R^2 + \underbrace{\text{---}\otimes\text{---}}_{\delta m^2, \delta Z} + \text{---}\text{---}\text{---} + \underbrace{\text{---}\otimes\text{---}}_{\delta \lambda} \right) G_F(x, y) = \int_0^{y^0} d^4 z \text{---}\text{---}\text{---} G_\rho(z, y) - \int_0^{x^0} d^4 z \text{---}\text{---}\text{---} G_F(z, y) - \int_{\mathbf{c}} d^4 z \text{---}\text{---}\text{---} \text{---}\text{---} G(z, y)$$

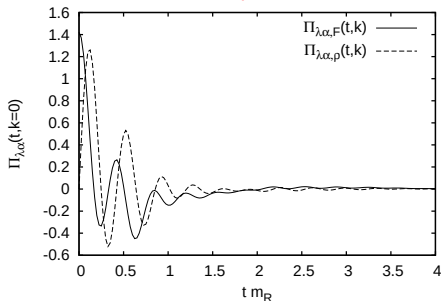
$\delta(z^0 - t_{\text{init}})$

$$\left(\partial_{x^0}^2 + \mathbf{k}^2 + m_R^2 + \underbrace{\text{---}\otimes\text{---}}_{\delta m^2, \delta Z} + \text{---}\text{---}\text{---} + \underbrace{\text{---}\otimes\text{---}}_{\delta \lambda} \right) G_\rho(x, y) = \int_{x^0}^{y^0} d^4 z \text{---}\text{---}\text{---} G_\rho(z, y)$$

Non-Gaussian contribution
on the right-hand side:

$$\int_{\mathbf{c}} d^4 z \Pi_{\lambda\alpha}(x, z) G(z, y) = \int d^3 z \Pi_{\lambda\alpha}(x, \mathbf{z}) G(z^0 = 0, \mathbf{z}, y)$$

e.g. Danielewicz, Semkat, Kremp, Bonitz, ...



UV divergences, renormalization and non-Gaussian ICs

Correlation energy at initial time

$$E_{kin}(t=0) = \frac{1}{2} \left[\partial_{x^0} \partial_{y^0} + \nabla^2 + m_R^2 + \underbrace{\text{---} \bigotimes \text{---}}_{\delta m^2, \delta Z} + \text{---} \bullet \text{---} + \underbrace{\text{---} \bigotimes \text{---}}_{\delta \lambda} \right] \text{---} \bullet |_{x=0} + \delta \Lambda$$

$$\begin{aligned}
 E_{corr}^{4-p.}(t=0) &= \frac{i}{4} \int_C d^4z \left[\Pi_{Gauss}(x, z) + \Pi_{non-Gauss}(x, z) \right] G(z, x) \Big|_{x=0} \\
 &= \underbrace{\int_0^t dz^0}_{\rightarrow 0} \left(\text{diagram 1} + \text{diagram 2} \right) \Big|_{x=0} \\
 &= \text{diagram 3} \Big|_{x=0}
 \end{aligned}$$

UV divergences, renormalization and non-Gaussian ICs

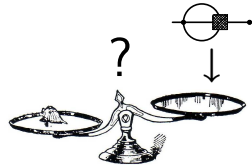
Correlation energy at initial time

$$E_{kin}(t=0) = \frac{1}{2} \left[\partial_{x^0} \partial_{y^0} + \nabla^2 + m_R^2 + \underbrace{\text{---} \otimes \text{---}}_{\delta m^2, \delta Z} + \text{---} \bullet \text{---} + \underbrace{\text{---} \otimes \text{---}}_{\delta \lambda} \right] \text{---} \bullet \Big|_{x=0} + \delta \Lambda$$

$$E_{corr}^{4-p.}(t=0) = \frac{i}{4} \int_{\mathcal{C}} d^4 z [\Pi_{Gauss}(x, z) + \Pi_{non-Gauss}(x, z)] G(z, x) \Big|_{x=0}$$

$$= \underbrace{\int_0^t dz^0 \text{---} \bullet \text{---} \bigcirc \text{---} \bullet \text{---} \Big|_{x=0}}_{\rightarrow 0} + \text{---} \bullet \text{---} \bigcirc \text{---} \text{---} \Big|_{x=0}$$

$$= \text{---} \bullet \text{---} \bigcirc \text{---} \text{---} \Big|_{x=0}$$



Questions

- Is it sufficient to include α_4 , or does one need α_6 etc. ?
- How to choose α_n ?

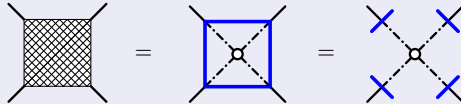
Renormalized initial state

$$\alpha_n(x_1, \dots, x_n) = \alpha_n^{vac/th}(x_1, \dots, x_n) + \Delta\alpha_n(x_1, \dots, x_n)$$

Thermal initial correlations

MG, Müller (2009)

$$\alpha_{4,3L}^{th,2PI}(z_1, z_2, z_3, z_4) = -i\lambda \int_{\mathcal{I}} d^4v \Delta(v, z_1) \Delta(v, z_2) \Delta(v, z_3) \Delta(v, z_4)$$



$$\begin{aligned} \Delta(-i\tau, z^0, \mathbf{k}) &= \text{---} \text{---} \text{---} | \text{---} \\ &= \Delta^+(-i\tau, \mathbf{k}) \delta_{\mathcal{C}}(z^0 - 0_+) + \Delta^-(-i\tau, \mathbf{k}) \delta_{\mathcal{C}}(z^0 - 0_-) \end{aligned}$$

$$\Delta^+(-i\tau, \mathbf{k}) = \frac{G_{th}^{\mathcal{II}}(-i\tau, 0, \mathbf{k})}{2G_{th}(0, 0, \mathbf{k})} + \partial_\tau G_{th}^{\mathcal{II}}(-i\tau, 0, \mathbf{k})$$

$$\Delta^-(-i\tau, \mathbf{k}) = \frac{G_{th}^{\mathcal{II}}(-i\tau, 0, \mathbf{k})}{2G_{th}(0, 0, \mathbf{k})} - \partial_\tau G_{th}^{\mathcal{II}}(-i\tau, 0, \mathbf{k})$$

UV divergences, renormalization and non-Gaussian ICs

Full **renormalized** thermal correlation energy (computed at initial time):

$$\begin{aligned}
 E_{corr}^{eq}(t=0) &= \frac{i}{4} \int \frac{d^4 z}{c} [\Pi_{Gauss}(x, z) + \Pi_{non-Gauss}(x, z)] G(z, x) \Big|_{x=0} \\
 &= \underbrace{\text{Diagram 1}}_{\int_0^{x^0} dz^0 \rightarrow 0} + \underbrace{\text{Diagram 2}}_{\int_0^{x^0} dz^0 \rightarrow 0} + \underbrace{\text{Diagram 3}}_{\int_0^{x^0} dz^0 \rightarrow 0} \\
 &\quad + \underbrace{\text{Diagram 4}}_{\int_0^{x^0} dz^0 \rightarrow 0} + \underbrace{\text{Diagram 5}}_{\int_0^{x^0} dz^0 \rightarrow 0} + \underbrace{\text{Diagram 6}}_{\int_0^{x^0} dz^0 \rightarrow 0} + \dots
 \end{aligned}$$

The diagrams represent Feynman diagrams for the thermal correlation energy. Each diagram consists of a horizontal line with a red dot at the left end (representing the initial time $x=0$). The diagrams are grouped into two rows. The first row contains three diagrams, and the second row contains three diagrams followed by an ellipsis. Each diagram is enclosed in a blue rectangular box. Below each diagram, a red integral $\int_0^{x^0} dz^0 \rightarrow 0$ is shown, indicating that the diagrams are UV divergent and require renormalization.

UV divergences, renormalization and non-Gaussian ICs

Full **renormalized** thermal correlation energy (computed at initial time):

$$\begin{aligned}
 E_{corr}^{eq}(t=0) &= \frac{i}{4} \int \frac{d^4 z}{c} [\Pi_{Gauss}(x, z) + \Pi_{non-Gauss}(x, z)] G(z, x) \Big|_{x=0} \\
 &= \underbrace{\text{Diagram 1}}_{\int_0^{x^0} dz^0 \rightarrow 0} + \underbrace{\text{Diagram 2}}_{\int_0^{x^0} dz^0 \rightarrow 0} + \underbrace{\text{Diagram 3}}_{\int_0^{x^0} dz^0 \rightarrow 0} \\
 &\quad + \underbrace{\text{Diagram 4}}_{\int_0^{x^0} dz^0 \rightarrow 0} + \underbrace{\text{Diagram 5}}_{\int_0^{x^0} dz^0 \rightarrow 0} + \underbrace{\text{Diagram 6}}_{\int_0^{x^0} dz^0 \rightarrow 0} + \dots \\
 &= \text{Diagram 7} \Big|_{x=0} = E_{4-p. corr}^{eq}(t=0)
 \end{aligned}$$

The diagrams represent Feynman diagrams for the thermal correlation energy. Diagram 1 is a self-energy loop. Diagrams 2-6 show higher-order corrections with multiple loops and vertices, each enclosed in a blue box. The red text below each diagram indicates that the integral $\int_0^{x^0} dz^0 \rightarrow 0$, which is a UV divergence. Diagram 7 is the final result, which is the 4-point correlation function $E_{4-p. corr}^{eq}(t=0)$.

...only the thermal **4-point** correlation contributes

⇒ truncate initial correlations with $n > 4$

Results for $\lambda\Phi^4$ 3-loop

Gaussian IC

$$G(x, y)|_{x^0, y^0=0} = G_{th}(x, y)|_{x^0, y^0=0}$$

$$\alpha_4(x_1, \dots, x_4) = 0$$

$$\alpha_n(x_1, \dots, x_n) = 0 \quad \text{for } n > 4$$

Non-Gaussian IC with α_4^{th}

$$G(x, y)|_{x^0, y^0=0} = G_{th}(x, y)|_{x^0, y^0=0}$$

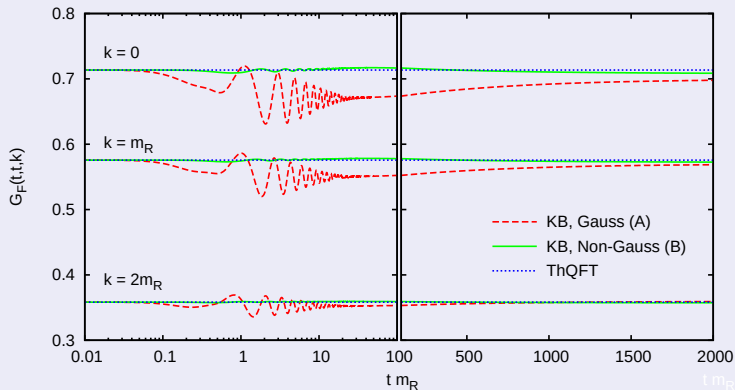
$$\alpha_4(x_1, \dots, x_4) = \alpha_4^{th}(x_1, \dots, x_4)$$

$$\alpha_n(x_1, \dots, x_n) = 0 \quad \text{for } n > 4$$

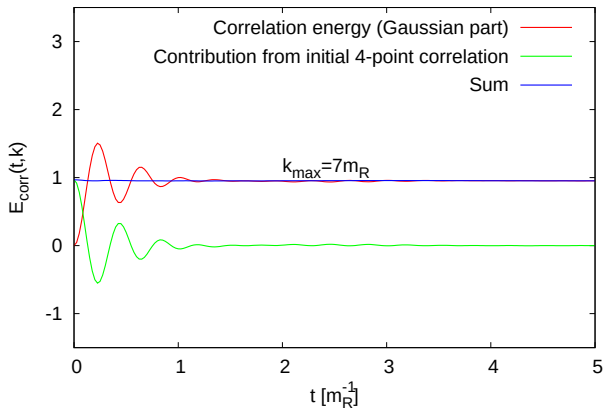
- Truncate thermal initial correlations
- $\Rightarrow$ *nonequilibrium* initial states
- The upper states are ‘as thermal as possible’
- Expectation: Non-Gaussian state more close to equilibrium

Minimal offset from thermal equilibrium

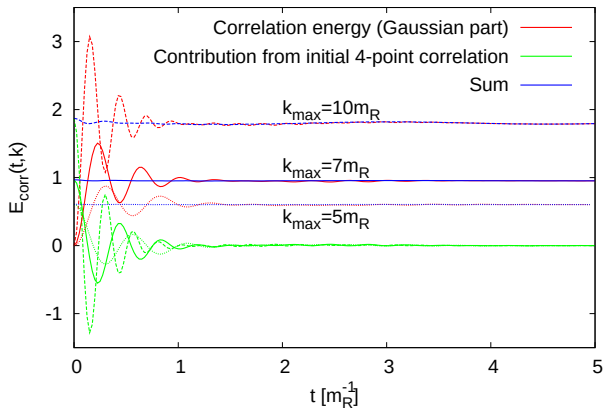
MG, Müller (2009)



Results for $\lambda\Phi^4$ 3-loop



Results for $\lambda\Phi^4$ 3-loop



Gaussian IC

$$\begin{aligned} E_{total} &= E_{kin}^{eq}(T_{init}) \\ &= E_{kin}^{eq}(T_{final}) + E_{corr}^{eq}(T_{final}) \end{aligned}$$

$\Rightarrow$ e.g. Köhler, Morawetz, ...

$$T_{init} \neq T_{final}$$

Cutoff-divergence $E_{corr}^{eq} \sim \Lambda^4 + T^2 \Lambda^2 + \dots$

$$|T_{init} - T_{final}| \sim \Lambda^2$$



Results for $\lambda\Phi^4$ 3-loop

Gaussian IC

$$\begin{aligned} E_{\text{total}} &= E_{\text{kin}}^{\text{eq}}(T_{\text{init}}) \\ &= E_{\text{kin}}^{\text{eq}}(T_{\text{final}}) + E_{\text{corr}}^{\text{eq}}(T_{\text{final}}) \end{aligned}$$

$\Rightarrow$ e.g. Köhler, Morawetz, ...

$$T_{\text{init}} \neq T_{\text{final}}$$

Cutoff-divergence $E_{\text{corr}}^{\text{eq}} \sim \Lambda^4 + T^2 \Lambda^2 + \dots$

$$|T_{\text{init}} - T_{\text{final}}| \sim \Lambda^2$$



Non-Gaussian IC with α_4^{th}

$$\begin{aligned} E_{\text{total}} &= E_{\text{kin}}^{\text{eq}}(T_{\text{init}}) + E_{4-p. \text{corr}}^{\text{eq}}(T_{\text{init}}) \\ &= E_{\text{kin}}^{\text{eq}}(T_{\text{final}}) + E_{\text{corr}}^{\text{eq}}(T_{\text{final}}) \end{aligned}$$

$$E_{4-p. \text{corr}}^{\text{eq}} = \text{diagram} = E_{\text{corr}}^{\text{eq}} \Rightarrow$$

$$T_{\text{init}} = T_{\text{final}}$$

No Cutoff-divergence

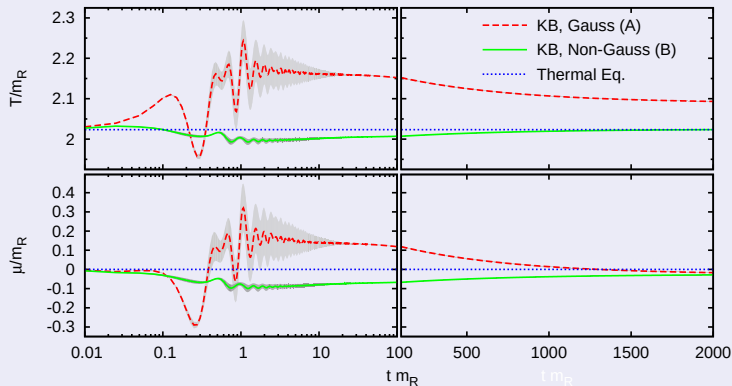
$$E_{\text{total}} = E^{\text{eq}}(T_{\text{init}}) = E^{\text{eq}}(T_{\text{final}}) = \text{finite}$$



Results for $\lambda\Phi^4$ 3-loop

No offset between initial and final temperature

MG, Müller (2009)

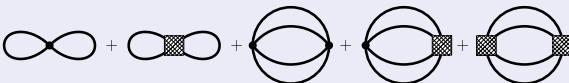


build-up kinetic - chemical equilibration

Renormalization of relativistic KBEs: status

MG, Müller (2009)

- It is necessary to include initial 4-point correlation α_4

$$\Gamma_2[G, \alpha_4] = \text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \text{diagram 4} + \text{diagram 5}$$


- Choosing $\alpha_4 = \alpha_4^{2PI, vac/th}$ renormalizes total (initial) energy

$$\text{diagram 1} = \text{diagram 2}$$

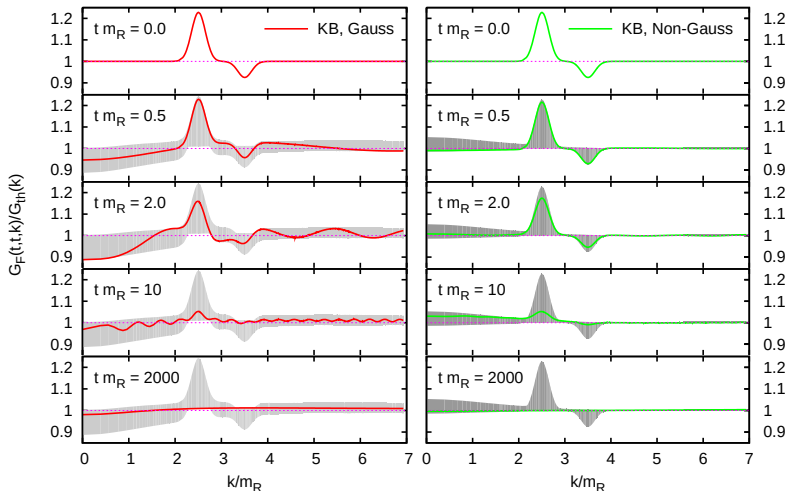

Restriction: Special class of ICs (close to equilibrium); $t \rightarrow 0, \infty$

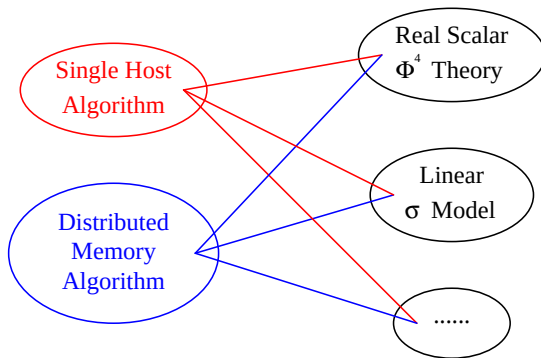
Questions

General conditions for renormalized initial states

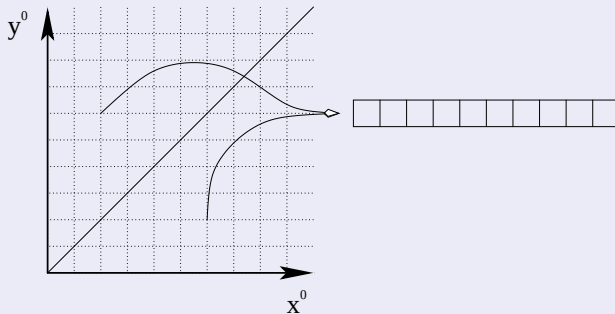
Results for $\lambda\Phi^4$ 3-loop

Example: $G(x^0, y^0, \mathbf{k})|_{x^0=y^0=t_{init}} = G_{vac/th}^{2PI}(\mathbf{k}) + \underbrace{\Delta G(\mathbf{k})}_{\rightarrow 0 \text{ for } \mathbf{k} \rightarrow \infty}, \quad \alpha_4 \in \{0, \alpha_4^{2PI, vac/th}\}$

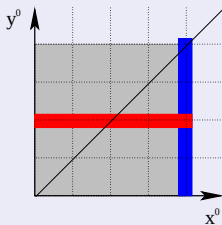




Single host algorithm: memory layout



Computation of self-energies and memory integrals



- Self-energy vector for $0 \leq y^0 \leq x^0$ using Fourier trf.

e.g. Danielewicz, Köhler, . . .

$$\Pi(x^0, y^0, \hat{\mathbf{k}}) = -\frac{\lambda^2}{6} \sum_{\hat{\mathbf{x}}} e^{i\hat{\mathbf{k}}\hat{\mathbf{x}}} \left[\sum_{\hat{\mathbf{p}}} e^{-i\hat{\mathbf{p}}\hat{\mathbf{x}}} G(x^0, y^0, \hat{\mathbf{p}}) \right]^3$$

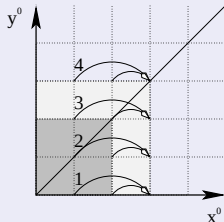
- Memory integrals = history matrix $\times$ self-energies

$$MEMINT(x^0, y^0, \hat{\mathbf{k}}) = \sum_{z^0} \Pi(x^0, z^0, \hat{\mathbf{k}}) G(z^0, y^0, \hat{\mathbf{k}})$$

time-stepping

$$\begin{aligned}\partial_{x^0}^2 G(x^0, y^0, k) &\rightarrow \Delta_0^b \Delta_0^f G(x^0, y^0, \hat{k}) \\ &= \frac{G(x^0 + a_t, y^0, \hat{k}) - 2G(x^0, y^0, \hat{k}) + G(x^0 - a_t, y^0, \hat{k})}{a_t^2}\end{aligned}$$

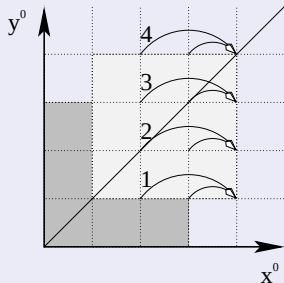
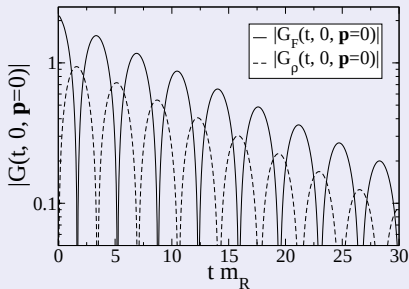
$$\begin{aligned}G(x^0 + a_t, y^0, \hat{k}) &= 2G(x^0, y^0, \hat{k}) - G(x^0 - a_t, y^0, \hat{k}) \\ &\quad + a_t^2 \left[MEMINT(x^0, y^0, \hat{k}) - (\hat{k}^2 + M^2(x^0)) G(x^0, y^0, \hat{k}) \right]\end{aligned}$$



Step 4: Use

$$\begin{aligned}G_F(x^0, y^0, \hat{k}) &= G_F(y^0, x^0, \hat{k}) \\ G_\rho(x^0, y^0, \hat{k}) &= -G_\rho(y^0, x^0, \hat{k})\end{aligned}$$

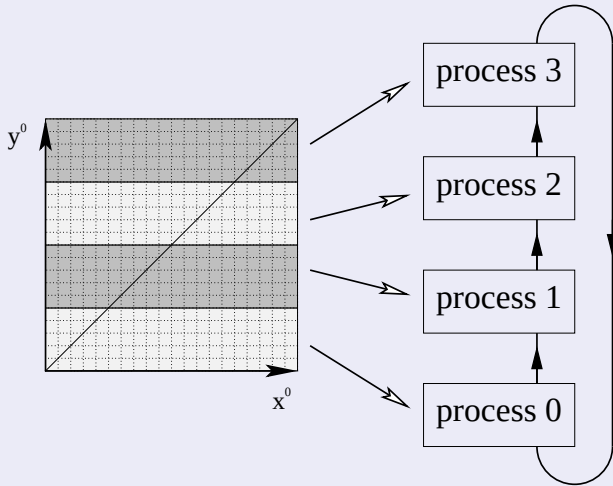
History cut-off



$$H = \{0, a_t, 2a_t, \dots, (N_t - 1) a_t\}^2$$

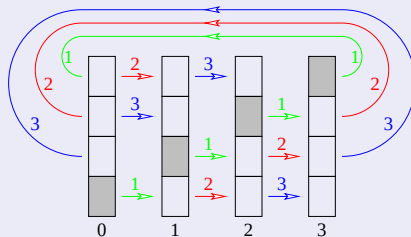
$$G(x^0, y^0, \hat{\mathbf{k}}) \rightarrow G(x^0 \bmod N_t, y^0 \bmod N_t, \hat{\mathbf{k}})$$

Parallelized distributed memory algorithm



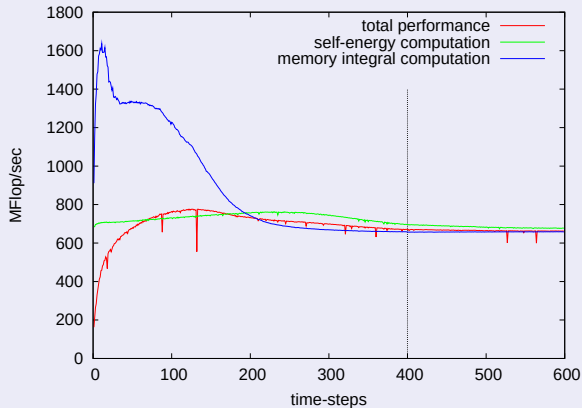
Parallelized distributed memory algorithm

- compute self-energies on each stripe in parallel
- circulate self-energies



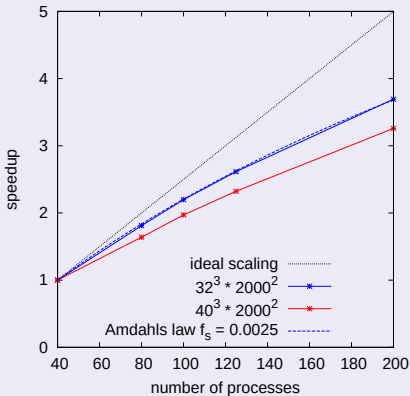
- compute memory integrals on each stripe in parallel
- time-stepping $(x^0, y^0) \rightarrow (x^0 + a_t, y^0)$ on each stripe in parallel
- mirror history matrix
- time-stepping $(x^0, x^0 + a_t) \rightarrow (x^0 + a_t, x^0 + a_t)$

Performance



Intel Itanium2 Montecito Dual Core 1.6GHz, Peak performance 6.4 GFlop/s per core

Scaling



$$s(N_p) = \frac{1}{f_s + \frac{f_p}{N_p} + k(N_p - 1)}$$

$$f_s : f_p \simeq 3 : 1000$$



LRZ, HLRB2 (SGI Altix 4700, 9728 cores, Total peak performance 62.3 TFlop/s)

Relativistic KBEs for correlated initial states and baryogenesis

thank you!



References

- MG, Markus Michael Müller, *Phys.Rev.D80:085011,2009*
- Borsanyi, Reinosa *Nucl.Phys.A820:147C,2009*
- MG, Markus Michael Müller, *proceedings of HLRB2 results workshop 2009*
- MG, Hohenegger, Kartavtsev, Lindner, *Phys.Rev.D80:125027,2009; arXiv:0911.4122; arXiv:1002.0331*

Thermal Initial Correlations

Thermal density matrix of the interacting theory

$$\rho_{th} = \frac{1}{Z} e^{-\beta H}, \quad H = H_0 + H_{int}$$

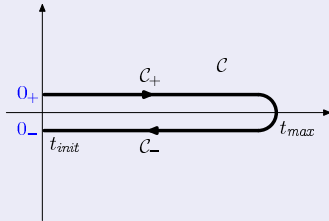
⇒ Compute the corresponding initial correlations

$$\langle \varphi_+ | \rho_{th} | \varphi_- \rangle = \exp \left(i \sum_{n=0}^{\infty} \int d^4 x_{12\dots n} \alpha_n^{th}(x_1, \dots, x_n) \varphi(x_1) \varphi(x_2) \cdots \varphi(x_n) \right)$$

- Can be done order-by-order in H_{int}
- Problem: Need approximations compatible with 2PI formalism
- Solution: Match 2PI on closed real-time path with 2PI thermal (imaginary-time) field theory *MG, Müller (2009)*

Thermal Initial Correlations

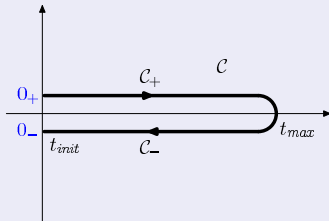
Closed time path $\mathcal{C}$ with α_n



$$\begin{aligned} & \langle \varphi_+ | \rho_{th} | \varphi_- \rangle \\ &= \exp \left(i \sum_{n=0}^{\infty} \alpha_{12 \dots n}^{th} \varphi_1 \varphi_2 \cdots \varphi_n \right) \end{aligned}$$

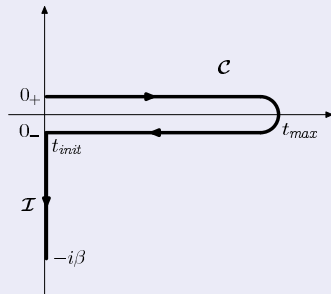
Thermal Initial Correlations

Closed time path $\mathcal{C}$ with α_n



$$\begin{aligned} & \langle \varphi_+ | \rho_{th} | \varphi_- \rangle \\ &= \exp \left(i \sum_{n=0}^{\infty} \alpha_{12 \dots n}^{th} \varphi_1 \varphi_2 \cdots \varphi_n \right) \end{aligned}$$

Thermal time path $\mathcal{C} + \mathcal{I}$



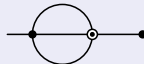
$$\begin{aligned} & \langle \varphi_+ | \rho_{th} | \varphi_- \rangle \\ &= \int_{\varphi(0, \mathbf{x}) = \varphi_-(\mathbf{x})}^{\varphi(-i\beta, \mathbf{x}) = \varphi_+(\mathbf{x})} \mathcal{D}\varphi \exp \left(i \int_{\mathcal{I}} d^4x \mathcal{L}(x) \right) \end{aligned}$$

Thermal time path $\mathcal{C} + \mathcal{I}$

Self-consistent Schwinger-Dyson equation

$$G_{th}^{-1}(x, y) = G_{0,th}^{-1}(x, y) - \Pi_{th}(x, y) \quad \Leftrightarrow$$

$$(\square_x + m^2)G_{th}(x, y) = -i\delta_{\mathcal{C}+\mathcal{I}}(x - y) - i \underbrace{\int_{\mathcal{C}+\mathcal{I}} d^4z \Pi_{th}(x, z) G_{th}(z, y)}_{\text{Diagram}}$$



Thermal Initial Correlations

Thermal time path $\mathcal{C} + \mathcal{I}$

Self-consistent Schwinger-Dyson equation

$$G_{th}^{-1}(x, y) = G_{0,th}^{-1}(x, y) - \Pi_{th}(x, y) \quad \Leftrightarrow$$

$$(\square_x + m^2) G_{th}(x, y) = -i\delta_{\mathcal{C}+\mathcal{I}}(x - y) - i \underbrace{\int_{\mathcal{C}+\mathcal{I}} d^4z \Pi_{th}(x, z) G_{th}(z, y)}_{\text{Diagram: a circle with a horizontal line through its center. The left end of the line has a solid black dot, and the right end has an open circle. The circle itself has a solid black dot on its right side. This represents a self-energy loop on a thermal propagator line.}}$$

Closed time path $\mathcal{C}$ with initial correlations α

Kadanoff-Baym equation for a Non-Gaussian initial state

$$\begin{aligned} (\square_x + m^2) G(x, y) &= -i\delta_{\mathcal{C}}(x - y) \\ &\quad - i \int_{\mathcal{C}} d^4z [\Pi_{Gauss}(x, z) + \Pi_{non-Gauss}(x, z)] G(z, y) \end{aligned}$$

Thermal Initial Correlations

Thermal time path $\mathcal{C} + \mathcal{I}$

The lines represent the *complete* propagator

$$\begin{aligned}
 G_{th}^{\mathcal{C}\mathcal{C}}(x, y) &= \bullet \text{---} \bullet & G_{th}^{\mathcal{C}\mathcal{I}}(x, y) &= \bullet \text{---} \circ \\
 G_{th}^{\mathcal{I}\mathcal{I}}(x, y) &= \circ \text{---} \circ & G_{th}^{\mathcal{I}\mathcal{C}}(x, y) &= \circ \text{---} \bullet \\
 -i\lambda \int_{\mathcal{C}} d^4x &= \text{X with } \bullet & -i\lambda \int_{\mathcal{I}} d^4x &= \text{X with } \circ & -i\lambda \int_{\mathcal{C}+\mathcal{I}} d^4x &= \text{X with } \odot
 \end{aligned}$$

Example: 2PI three-loop truncation

$$\text{Diagram 1} = \text{Diagram 2} + \text{Diagram 3}$$

Thermal Initial Correlations: Perturbation Theory

$$G_{0,th}^{\mathcal{I}\mathcal{C}}(-i\tau, y^0, \mathbf{k}) = \int_c dt \Delta_0(-i\tau, t, \mathbf{k}) G_{0,th}^{\mathcal{C}\mathcal{C}}(t, y^0, \mathbf{k})$$

$$\text{○} \text{---} \text{●} = \text{○} \cdots \text{+} \text{---} \text{●}$$

Free 'connection'

$$\begin{aligned} \Delta_0(-i\tau, z^0, \mathbf{k}) &= \Delta_0^+(-i\tau, \mathbf{k}) \delta_{\mathcal{C}}(z^0 - 0_+) + \Delta_0^-(-i\tau, \mathbf{k}) \delta_{\mathcal{C}}(z^0 - 0_-) \\ &= \cdots \cdots \text{+} \text{---} \end{aligned}$$

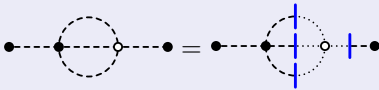
where

$$\begin{aligned} \Delta_0^+(-i\tau, \mathbf{k}) &= \frac{\sinh(\omega_{\mathbf{k}}\tau)}{\sinh(\omega_{\mathbf{k}}\beta)} = \frac{G_{0,th}^{\mathcal{I}\mathcal{I}}(-i\tau, 0, \mathbf{k})}{2G_{0,th}(0, 0, \mathbf{k})} + \partial_\tau G_{0,th}^{\mathcal{I}\mathcal{I}}(-i\tau, 0, \mathbf{k}) \\ \Delta_0^-(-i\tau, \mathbf{k}) &= \frac{\sinh(\omega_{\mathbf{k}}(\beta - \tau))}{\sinh(\omega_{\mathbf{k}}\beta)} = \frac{G_{0,th}^{\mathcal{I}\mathcal{I}}(-i\tau, 0, \mathbf{k})}{2G_{0,th}(0, 0, \mathbf{k})} - \partial_\tau G_{0,th}^{\mathcal{I}\mathcal{I}}(-i\tau, 0, \mathbf{k}) \end{aligned}$$

Thermal Initial Correlations: Perturbation Theory

Example

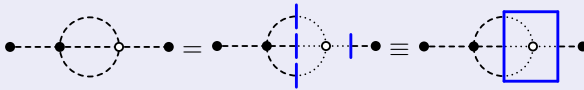
MG, Müller (2009)



Thermal Initial Correlations: Perturbation Theory

Example

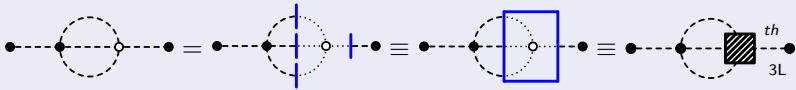
MG, Müller (2009)



Thermal Initial Correlations: Perturbation Theory

Example

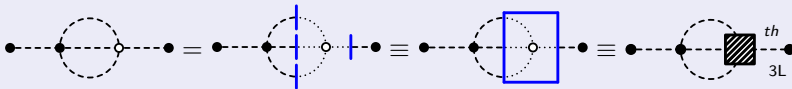
MG, Müller (2009)



Thermal Initial Correlations: Perturbation Theory

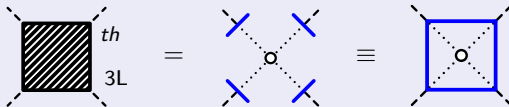
Example

MG, Müller (2009)



Perturbative initial 4-point correlation

$$\alpha_{4,3L}^{th}(\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \mathbf{z}_4) = -i\lambda \int_{\mathcal{I}} d^4v \Delta_0(v, \mathbf{z}_1) \Delta_0(v, \mathbf{z}_2) \Delta_0(v, \mathbf{z}_3) \Delta_0(v, \mathbf{z}_4)$$



Thermal Initial Correlations: 2PI

$$G_{th}^{\mathcal{IC}}(-i\tau, \textcolor{red}{y}^0, \mathbf{k}) = \int_{\mathcal{C}} d\textcolor{red}{t} \tilde{\Delta}(-i\tau, \textcolor{red}{t}, \mathbf{k}) G_{th}^{\mathcal{CC}}(\textcolor{red}{t}, \textcolor{red}{y}^0, \mathbf{k})$$

$$\text{---} \circ \text{---} \bullet = \text{---} \circ \text{---} \text{---} \bullet$$

Complete 'connection':

$$\tilde{\Delta}(-i\tau, z^0, \mathbf{k}) = \text{---}\cdot\cdots\cdot\text{---}\perp\text{---} = \text{---}\cdot\cdots\cdot\text{---}\color{blue}{|}\text{---} + \text{---}\cdot\cdots\cdot\text{---}\circ\Pi_{th}^{nl}\text{---}$$

$$\begin{aligned}\Delta(-i\tau, z^0, \mathbf{k}) &= \text{---} \cdots \cdots \text{---} | \text{---} \\ &= \Delta^+(-i\tau, \mathbf{k}) \delta_{\mathcal{C}}(z^0 - 0_+) + \Delta^-(-i\tau, \mathbf{k}) \delta_{\mathcal{C}}(z^0 - 0_-)\end{aligned}$$

where

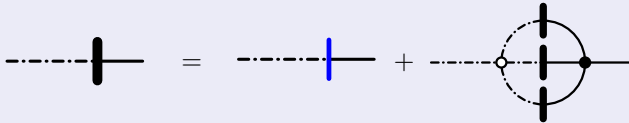
$$\Delta^+(-i\tau, \mathbf{k}) = \frac{G_{th}^{\mathcal{I}\mathcal{I}}(-i\tau, 0, \mathbf{k})}{2G_{th}(0, 0, \mathbf{k})} + \partial_\tau G_{th}^{\mathcal{I}\mathcal{I}}(-i\tau, 0, \mathbf{k})$$

$$\Delta^{-}(-i\tau, \mathbf{k}) = \frac{G_{th}^{\mathcal{II}}(-i\tau, 0, \mathbf{k})}{2G_{th}(0, 0, \mathbf{k})} - \partial_{\tau} G_{th}^{\mathcal{II}}(-i\tau, 0, \mathbf{k})$$

Thermal Initial Correlations: 2PI

Example: 2PI three-loop truncation

MG, Müller (2009)



Thermal Initial Correlations: 2PI

Example: 2PI three-loop truncation

MG, Müller (2009)

A diagrammatic equation representing the 2PI three-loop truncation. On the left, a horizontal dashed line is terminated by a thick black vertical bar. This is equal to the sum of two terms. The first term is a horizontal dashed line terminated by a blue vertical bar. The second term is a horizontal dashed line connected to a circle. Inside the circle, there are three thick black vertical bars. The circle has a white dot on its left edge and a black dot on its right edge, with a horizontal line passing through them.

Iterative Solution:

A diagrammatic equation representing the iterative solution. On the left, a horizontal dashed line is terminated by a thick black vertical bar. This is equal to the sum of two terms. The first term is a horizontal dashed line terminated by a blue vertical bar. The second term is a horizontal dashed line connected to a circle. Inside the circle, there are three blue vertical bars. The circle has a white dot on its left edge and a black dot on its right edge, with a horizontal line passing through them.

Thermal Initial Correlations: 2PI

Example: 2PI three-loop truncation

MG, Müller (2009)

A diagrammatic equation representing the 2PI three-loop truncation. On the left, a horizontal dashed line is terminated by a thick black vertical bar. This is equal to the sum of two terms. The first term is a horizontal dashed line terminated by a blue vertical bar. The second term is a horizontal dashed line entering a circle from the left, with a small white dot at the entry point. Inside the circle, there is a thick black vertical bar. A solid black line exits the circle to the right, ending at a black dot. A dashed circle is also centered within the main circle, passing through the entry and exit points.

Iterative Solution:

A diagrammatic equation showing the iterative solution. The first line shows a horizontal dashed line terminated by a thick black vertical bar, equal to a horizontal dashed line terminated by a blue vertical bar, plus a diagram of a circle with a dashed line entering from the left (white dot), a thick black vertical bar inside, and a solid line exiting to the right (black dot), with an internal dashed circle. Below this, a plus sign is followed by a more complex diagram: a circle with a dashed line entering from the left (white dot), a blue vertical bar inside, and a solid line exiting to the right (black dot). Inside this circle is another circle, which contains a smaller circle with a dashed line entering from the left (white dot) and a solid line exiting to the right (black dot). A thick black vertical bar is also present inside the innermost circle.

Thermal Initial Correlations: 2PI

Example: 2PI three-loop truncation

MG, Müller (2009)

A diagrammatic equation representing the 2PI three-loop truncation. On the left, a horizontal dashed line is terminated by a thick vertical black bar. This is equal to the sum of two terms. The first term is a horizontal dashed line terminated by a thick vertical blue bar. The second term is a horizontal dashed line entering a circle from the left, with a thick vertical black bar inside the circle, and a horizontal solid line exiting the circle to the right.

Iterative Solution:

A diagrammatic equation representing the iterative solution. The left side is identical to the first equation: a horizontal dashed line with a thick vertical black bar. This is equal to the sum of three terms. The first term is a horizontal dashed line with a thick vertical blue bar. The second term is a horizontal dashed line entering a circle with a thick vertical blue bar inside, and a horizontal solid line exiting. The third term is a horizontal dashed line entering a circle with a thick vertical blue bar inside, and a horizontal solid line exiting; this circle contains a smaller circle with a thick vertical blue bar inside it.

Thermal Initial Correlations: 2PI

Example: 2PI three-loop truncation

MG, Müller (2009)

A diagrammatic equation showing the truncation of the 2PI hierarchy. On the left, a thick vertical black line is connected to a horizontal dashed line. This is equal to the sum of two terms: a horizontal dashed line connected to a thin vertical blue line, and a diagram consisting of a horizontal dashed line connected to a circle. Inside the circle, there is a thick vertical black line and a thin vertical blue line, with a solid black dot on the right side of the circle.

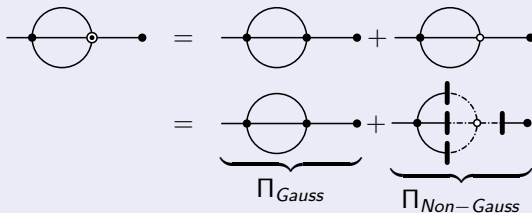
Iterative Solution:

An iterative solution diagram showing a series of terms. The first row shows the thick vertical black line equal to the sum of the thin vertical blue line and the first diagram (a circle with a thick vertical black line, a thin vertical blue line, and a solid black dot on the right). The second row continues the series with three more diagrams: a circle with two concentric circles (the inner one has a thick vertical black line and the outer one has a thin vertical blue line), a circle with two smaller circles inside (each having a thick vertical black line and a thin vertical blue line), and a circle with three even smaller circles inside (each having a thick vertical black line and a thin vertical blue line). The series ends with a plus sign and an ellipsis.

Thermal Initial Correlations: 2PI

Example: 2PI three-loop truncation

MG, Müller (2009)



The diagram illustrates a truncation scheme for the 2PI effective action. It shows a diagrammatic equation where a self-energy diagram (a circle with a horizontal line through its center, with a solid dot on the left and an open circle on the right) is equal to the sum of two terms. The first term is a self-energy diagram with two solid dots on the horizontal line. The second term is a self-energy diagram with two solid dots on the horizontal line and a dashed circle with two vertical lines through its center. Below the first term is a brace labeled Π_{Gauss} , and below the second term is a brace labeled $\Pi_{Non-Gauss}$.

$$\text{Diagram} = \text{Diagram} + \text{Diagram}$$
$$= \underbrace{\text{Diagram}}_{\Pi_{Gauss}} + \underbrace{\text{Diagram}}_{\Pi_{Non-Gauss}}$$

Example: 2PI three-loop truncation

MG, Müller (2009)

$$\begin{aligned}
 \text{Self-energy diagram} &= \text{Bubble diagram} + \text{Tadpole diagram} \\
 &= \underbrace{\text{Bubble diagram}}_{\Pi_{Gauss}} + \underbrace{\text{Tadpole diagram}}_{\Pi_{Non-Gauss}}
 \end{aligned}$$

Non-Gaussian self-energy contains $\alpha_n^{th}(x_1, \dots, x_n)$ for all $n \geq 4$

$$\Pi_{non-Gauss}(x, z) = \text{Diagram with a blue square box around the tadpole part}$$

Example: 2PI three-loop truncation

MG, Müller (2009)

$$\begin{aligned}
 \text{Self-energy diagram} &= \text{Bubble diagram} + \text{Tadpole diagram} \\
 &= \underbrace{\text{Bubble diagram}}_{\Pi_{Gauss}} + \underbrace{\text{Tadpole diagram}}_{\Pi_{Non-Gauss}}
 \end{aligned}$$

Non-Gaussian self-energy contains $\alpha_n^{th}(x_1, \dots, x_n)$ for all $n \geq 4$

$$\Pi_{non-Gauss}(x, z) = \text{Bubble diagram with box} + \text{Tadpole diagram with box}$$

Thermal Initial Correlations: 2PI

Example: 2PI three-loop truncation

MG, Müller (2009)

$$\begin{aligned}
 \text{Self-energy diagram} &= \text{Bubble diagram} + \text{Tadpole diagram} \\
 &= \underbrace{\text{Bubble diagram}}_{\Pi_{Gauss}} + \underbrace{\text{Tadpole diagram}}_{\Pi_{Non-Gauss}}
 \end{aligned}$$

Non-Gaussian self-energy contains $\alpha_n^{th}(x_1, \dots, x_n)$ for all $n \geq 4$

$$\Pi_{non-Gauss}(x, z) = \text{Bubble with 1 vertical line} + \text{Bubble with 2 vertical lines} + \text{Bubble with 3 vertical lines}$$

Thermal Initial Correlations: 2PI

Example: 2PI three-loop truncation

MG, Müller (2009)

$$\begin{aligned}
 \text{Self-energy diagram} &= \text{Bubble diagram} + \text{Tadpole diagram} \\
 &= \underbrace{\text{Bubble diagram}}_{\Pi_{Gauss}} + \underbrace{\text{Tadpole diagram}}_{\Pi_{Non-Gauss}}
 \end{aligned}$$

Non-Gaussian self-energy contains $\alpha_n^{th}(x_1, \dots, x_n)$ for all $n \geq 4$

$$\Pi_{non-Gauss}(x, z) = \text{Bubble diagram with vertical line} + \text{Bubble diagram with vertical and horizontal lines} + \text{Bubble diagram with vertical and two horizontal lines} + \text{Tadpole diagram with vertical and two horizontal lines}$$

Thermal Initial Correlations: 2PI

Example: 2PI three-loop truncation

MG, Müller (2009)

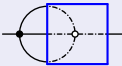
$$\begin{aligned}
 \text{Self-energy diagram} &= \text{Bubble diagram} + \text{Tadpole diagram} \\
 &= \underbrace{\text{Bubble diagram}}_{\Pi_{Gauss}} + \underbrace{\text{Tadpole diagram}}_{\Pi_{Non-Gauss}}
 \end{aligned}$$

Non-Gaussian self-energy contains $\alpha_n^{th}(x_1, \dots, x_n)$ for all $n \geq 4$

$$\begin{aligned}
 \Pi_{non-Gauss}(x, z) &= \text{Bubble diagram with vertical line} + \text{Bubble diagram with horizontal line} + \text{Bubble diagram with vertical and horizontal lines} \\
 &+ \text{Bubble diagram with vertical and horizontal lines} + \text{Bubble diagram with vertical and horizontal lines} + \dots
 \end{aligned}$$

Result: Example for Setting-Sun Approximation

$$\Pi_{Gauss} = \text{---}\bullet\text{---} \overset{\text{loop}}{\cap} \text{---}\bullet\text{---} + \text{---}\bullet\text{---} \bigcirc \text{---}\bullet\text{---}$$

$$\Pi_{non-Gauss}^{th} = \text{---}\bullet\text{---} \bigcirc \text{---}\bullet\text{---}$$
A Feynman diagram representing a non-Gaussian approximation. It consists of a horizontal line with two vertices (black dots). A loop is attached to the right vertex. The loop is a circle with a vertical dashed line passing through its center. A blue rectangular box is drawn around the right half of the loop, from the vertical dashed line to the right edge of the circle.

Result: Example for Setting-Sun Approximation

$$\Pi_{Gauss} = \text{---} \bullet \text{---} \text{---} + \text{---} \bullet \text{---} \text{---}$$

$$\Pi_{non-Gauss}^{th} = \text{---} \bullet \text{---} \text{---} + \text{---} \bullet \text{---} \text{---}$$

UV divergences, renormalization and non-Gaussian ICs

Result: Example for Setting-Sun Approximation

$$\Pi_{Gauss} = \text{---} \overset{\text{loop}}{\bullet} \text{---} + \text{---} \bullet \text{---} \text{---} \bullet \text{---}$$

$$\Pi_{non-Gauss}^{th} = \text{---} \bullet \text{---} \text{---} \bullet \text{---} + \text{---} \bullet \text{---} \text{---} \bullet \text{---} \text{---} \bullet \text{---} + \text{---} \bullet \text{---} \text{---} \bullet \text{---} \text{---} \bullet \text{---} \text{---} \bullet \text{---}$$

Result: Example for Setting-Sun Approximation

$$\Pi_{Gauss} = \text{---} \bullet \text{---} \text{---} + \text{---} \bullet \text{---} \text{---}$$

$$\Pi_{non-Gauss}^{th} = \text{---} \bullet \text{---} \text{---} + \text{---} \bullet \text{---} \text{---} + \text{---} \bullet \text{---} \text{---} + \text{---} \bullet \text{---} \text{---}$$

Result: Example for Setting-Sun Approximation

$$\Pi_{Gauss} = \text{---} \bullet \text{---} \text{---} + \text{---} \bullet \text{---} \bullet \text{---}$$

$$\Pi_{non-Gauss}^{th} = \text{---} \bullet \text{---} \text{---} + \text{---} \bullet \text{---} \text{---} + \text{---} \bullet \text{---} \text{---} + \text{---} \bullet \text{---} \text{---} + \text{---} \bullet \text{---} \text{---} + \dots$$

Result: Example for Setting-Sun Approximation

$$\Pi_{Gauss} = \text{---} \bullet \text{---} + \text{---} \bullet \text{---} \bullet \text{---}$$

$$\Pi_{non-Gauss}^{th} = \text{---} \bullet \text{---} \bullet \text{---} + \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} + \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} + \dots$$

The diagrams for $\Pi_{non-Gauss}^{th}$ show a series of higher-order corrections to the Gaussian approximation. Each diagram consists of a horizontal line with vertices (black dots) and internal loops (solid and dashed lines). Blue boxes highlight the internal loop structures, which represent thermal initial correlations. The series includes terms with one, two, and three internal loops, followed by an ellipsis indicating higher-order terms.

Kadanoff-Baym equations with thermal initial correlations contain

$$\alpha_n^{th}(x_1, \dots, x_n) \quad \text{for all } n \geq 4$$