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Evolution of the Entropy of Stationary States in Selforganization Processes in the Control Parameter Space

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The behaviour of entropy (Shannon-information) and renormalized entropy (based on the S-theorem [3]) is investigated for systems with an exponential stationary probability distribution function (1). Analytical results for the derivatives with respect to the control parameters are derived. One class of systems (3) is separated for which the renormalized entropy is a monotonously decreasing function of the control parameters.

1. Introduction

At the present moment two criteria for the relative degree of order or the contents of information respectively of stationary states in selforganizing systems [1, 2] are proposed.

One is based on the S-theorem which was stated first for the generation of self-sustained oscillations in a Van der Pol system [3] and for the transition from a laminar to a turbulent flow [4]. The advantage of this approach was then demonstrated for a number of other examples in [5–14]. In [15] the S-theorem was presented in a more general form.

The second criterion was offered in recent papers by H. Haken [16, 17]. He used the S-information.

The origin of the letter "S" in "S-theorem" and "S-information" is different. In the first it symbolizes the role of selforganization and synergetics, however in the second case it is a regard for one of the pioneers of information theory C. Shannon.

In the present paper we provide some further results for the approach to the degree of order of stationary states in synergetics that uses the S-theorem. We focus on a class of systems (1), that can be simply analyzed with analytical methods. We compare the results for the degree of order provided by both criteria.

For a concrete example of a selforganizing system with two control parameters we show that in the presence of a second parameter the system can reach more ordered states, i.e. states with lower entropy.

2. Exponential Stationary Probability Distribution Function

Let us study stationary probability distribution functions (S.p.d.f.) of the following form [18]:

$$f(X, \mathbf{a}) = \exp((F(\mathbf{a}) - H(X, \mathbf{a}))/D_0),$$
$$\int f(X, \mathbf{a})\,\mathrm{d}X = 1. \qquad (1)$$

Here D_0 is the constant intensity of a noise source or the temperature in a thermostat system respectively. X is a set of order parameters or a complete set of phase variables respectively, $\mathbf{a} = (a_1, \ldots, a_n)$ are the control parameters characterizing external forces. We always can define them so, that they vanish in equilibrium, i.e. in the state without external excitation (cf. [18]). $F(\mathbf{a})$ is the normalization constant. From $H(X, \mathbf{a})$ we separate the effective hamiltonian, i.e. the equilibrium part:

$$H(X, \mathbf{a}) = H_0(X) - U(X, \mathbf{a}), \qquad H_0(X) = H(X, 0). \qquad (2)$$

Stationary probability distribution functions of the form (1) play a very important role in statistical physics, e.g. Gibbs-distributions in equilibrium, local Gibbs-distributions in the nonequilibrium situation [19], but also in the theory of selforganization, e.g. stationary solutions of Fokker-Planck equations in the case of detailed balance and so on.

In the following we investigate how the function (1) and the mean values of certain quantities, especially the S-information change if the control parameters alter (Sects. 3 and 4). In Sects. 5 and 6 we study the behaviour of the entropy of states which are renormalized to a given constant value of the mean effective energy. We find general formulae which we shall analyze in more detail for one widespread case:

$$U(X, \mathbf{a}) = \sum_{i=1}^{n} g_i(a_i)\, v_i(X). \tag{3}$$

3. The Case of a Single Control Parameter

We can study the control parameter dependence of the function (1) by differentiation with respect to "a":

$$\frac{\partial}{\partial a} f(X, a) = \frac{1}{D_0}\left(\frac{\partial}{\partial a} U(X, a) + F'(a)\right) f(X, a). \tag{4}$$

Using the normalization condition $F'(a)$ can be expressed by $U(X, a)$:

$$F'(a) = -\left\langle \frac{\partial}{\partial a} U \right\rangle.$$

The brackets denote the averaging over the function (1). Now we can write (4) in the form (5):

$$\frac{\partial}{\partial a} f(X, a) = \frac{1}{D_0} f(X, a)\, \Delta \frac{\partial}{\partial a} U(X, a). \tag{5}$$

The symbol "Δ" denotes the deviation from the mean value (fluctuation):

$$\Delta \frac{\partial}{\partial a} U = \frac{\partial}{\partial a} U - \left\langle \frac{\partial}{\partial a} U \right\rangle.$$

We now consider an arbitrary function $A(X)$.

$$\frac{\mathrm{d}}{\mathrm{d}a} \langle A \rangle = \int A(X) \frac{\partial}{\partial a} f(X, a)\, \mathrm{d}X.$$

Making use of (5) we can write

$$\frac{\mathrm{d}}{\mathrm{d}a} \langle A \rangle = \frac{1}{D_0} \left\langle A \Delta \frac{\partial}{\partial a} U \right\rangle. \tag{6}$$

Of special interest is the sign of this expression. It depends on the character of the external excitation, i.e. on $U(X, a)$. We demonstrate this on the simplest example.

Let f and A be functions of a single positive variable (amplitude, energy, concentration and so on), $0 \leqq E \leqq \infty$. If $A(E)$ is a monotonous function, there exists a simple condition for the sign of (6):

$$\frac{\mathrm{d}}{\mathrm{d}a} \langle A \rangle \frac{\mathrm{d}}{\mathrm{d}E} A \geqq 0 \quad (\leqq 0) \tag{7}$$

if with increasing "a" the function $f(E, a)$ shifts to the right (left). The latter means in mathematical terms, that there exists a value $E = E_1$, that for all $E > E_1$: $\frac{\partial}{\partial a} f(E, a) \geqq 0$ ($\leqq 0$).

The condition (7) holds also for certain nonmonotonous functions $A(E)$ (see Appendix).

4. The Behaviour of the S-Information (Entropy)

We start from Shannon-information

$$S(a) = -\int f(X, a) \ln f(X, a)\, \mathrm{d}X.$$

Differentiation with respect to the control parameter provides

$$\frac{\mathrm{d}}{\mathrm{d}a} S(a) = -\int (1 + \ln f(X, a)) \frac{\partial}{\partial a} f(X, a)\, \mathrm{d}X.$$

We take into account (5) and also:

$$\Delta \ln f(X, a) = (\Delta U(X, a) - \Delta H_0(X))/D_0,$$

$$\frac{\mathrm{d}}{\mathrm{d}a} S(a) = -\frac{1}{D_0^2} \left\langle U \Delta \frac{\partial U}{\partial a} \right\rangle + \frac{1}{D_0^2} \left\langle H_0 \Delta \frac{\partial U}{\partial a} \right\rangle. \tag{8}$$

Let us first apply (8) to an example, which had been studied recently by Haken [16]:

$$f(q) = N \exp(\alpha q^2 - \beta q^4). \tag{9}$$

(9) is the order parameter distribution function undergoing a second order nonequilibrium phase transition (at $\alpha = 0$) from a monostable to a bistable state. β is positive, α we divide into two positive parts $-\alpha = \gamma - a_p$, γ is a system constant, a_p is the control parameter (pumping). The value of "a_p" defines the deviation from equilibrium (in equilibrium the pumping vanishes). Then (9) has the form (1, 2), where $H_0(q) = \gamma q^2 + \beta q^4$, $U(q, a_p) = a_p q^2$, $D_0 = 1$. Introducing H_0, U, D_o into (8) we find promptly

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}a_p} S &= \frac{\mathrm{d}}{\mathrm{d}\alpha} S \\ &= -\alpha \langle q^4 \rangle + \alpha \langle q^2 \rangle^2 + \beta \langle q^6 \rangle - \beta \langle q^2 \rangle \langle q^4 \rangle \\ &= -\alpha \sigma_{q^2}^2 + \beta \langle \Delta q^2\, \Delta q^4 \rangle \end{aligned} \tag{10}$$

σ^2 denotes the variance: $\sigma_{q^2}^2 = \langle (\Delta q^2)^2 \rangle$ (10) agrees with the result of [16].

Using the criterion (7) we can define the sign of the correlator:

$\langle \Delta q^2 \Delta q^4 \rangle = \langle q^2 \Delta q^4 \rangle > 0.$

The first term in (10) is positive in the monostable state ($\alpha < 0$) and negative in the bistable ($\alpha > 0$). The S-information has a single maximum near the transition point [17]. However it is doubtful, if this hump is caused by critical fluctuations that are typical for phase transitions.

It is enough to look at another nonequilibrium phase transition [18]:

$$f(E) = N \exp(-(\gamma E - a_p E + b/2 E^2)/D_0),$$
$$\gamma, a_p, b_0 \geqq 0, \quad H_0(E) = \gamma E + b/2 E^2,$$
$$U(E, a_p) = a_p E. \tag{11}$$

The system (11) is undergoing a Hopf-bifurcation at $a_p = \gamma$, e.g. onset of selfsustained oscillations and so on. It is interesting that the S-information is in this case a monotonously increasing function of the feedback "a_p" in the whole region, cf. [20, 21].

Let us come back to the general formula (8) which can be expressed in another form

$$\frac{d}{da} S = \frac{d}{da} S_1 + \frac{1}{D_0} \frac{d}{da} \langle H_0 \rangle,$$
$$S_1 = S - D_0 \langle H_0 \rangle = -\frac{1}{D_0} \phi, \tag{12}$$

ϕ is the nonequilibrium analogue of the free energy. In equilibrium, i.e. $U = 0$ we have $F_0 = H_0 + D_0 \ln f_0$, cf. (1, 2).

S_1 is connected with the information gain

$$K = \int dX f \ln f/f_0 = (\phi - F_0)/D_0,$$
$$\frac{d}{da} S_1 = -1/D_0 \frac{d}{da} \phi = -1/D_0^2 \left\langle U \Delta \frac{\partial}{\partial a} U \right\rangle. \tag{13}$$

In selforganization processes in systems (1, 2), as a rule, we expect an increase of the mean effective energy, i.e. $\frac{d}{da} \langle H_0 \rangle > 0$, if "$a$" measures the distance from equilibrium. The entropy S_1 is monotonously decreasing with increasing absolute values of the pumping if the condition (3) is fulfilled:

$$\frac{d}{da} S_1(a) = -\frac{1}{D_0^2} g(a) g'(a) \sigma_v^2 \leqq 0,$$
$$\text{if } \frac{d}{da} g^2(a) \geqq 0. \tag{13*}$$

Thus in this case the S-information is a monotonously increasing function of the control parameter for negative or small positive values of "a". The sign of the entropy derivative for higher values of "a" depends on the competition of the two terms in (12).

5. Renormalization of the Entropy

The entropy can serve for measuring the order (or chaoticy) in open systems only if it is calculated at fixed mean energy values of the states [11–15, 18].

In the following we study the behaviour of entropy under an additional condition $\langle W \rangle = W_0 = \text{const}$. Let W be an arbitrary quantity of state, which later will be replaced by the effective hamiltonian $H_0(X)$. Its mean value shall depend on system parameters $\gamma = \gamma_1, \ldots, \gamma_m$, on one control parameter "a" and on the intensity of external noise D_0. Of course, in general, $\frac{\partial}{\partial a} \langle W \rangle \neq 0$.

Then the condition $\langle W \rangle (\gamma, a, D_0) = W_0 = \text{const}$ can be understood as an equation for an implicite function $\tilde{D} = \tilde{D}(\gamma, a, W_0)$. The renormalized noise intensity (or the temperature if we deal with a system in a thermostat respectively) has to be real and positive. Moreover it should guarantee that the renormalized s.p.d.f. (see below) can also be normalized [11]. General conditions of the existence and uniqueness of such functions are, however, still unknown. There exist two approaches to the renormalization:

First way [3, 8, 9, 18, 20–22].

We replace D_0 with $\tilde{D}(\gamma, a, W_0)$ in the s.p.d.f. (1, 2), (below we will drop the arguments γ and W_0 in $\tilde{D}$)

$$\tilde{f}(X, a, W_0) = f(X, a)|_{D_0 \to \tilde{D}(a)}$$
$$= \exp((\tilde{F}(a) - H_0(X) + U(X, a))/\tilde{D}(a)). \tag{14}$$

The function (14) fulfills two conditions:

$$\int \tilde{f}(X, a, W_0)\, dX$$
$$= \int \tilde{f}(X, 0, W_0) dX = \int f_0(X)\, dX = 1, \tag{15}$$
$$\int W(X) \tilde{f}(X, a, W_0)\, dX$$
$$= \int W(X) f_0(X)\, dX = W_0 = \text{const}. \tag{16}$$

Thus we find the entropy of states at the condition $\langle W \rangle = \text{const}$:

$$\tilde{S}(a)|_{\langle W \rangle = W_0} = \tilde{S}(a, W_0)$$
$$= -\int \tilde{f}(X, a, W_0) \ln \tilde{f}(X, a, W_0)\, dX.$$

Naturally, $\langle W \rangle$ can also be fixed on any other level.

Second way [11, 12, 15].

Of course, we can renormalize the equilibrium p.d.f. (1, 2, with $U = 0$) instead of the nonequilibrium function:

$$\tilde{f}_0(X, a) = f_0(X)|_{D_0 \to \tilde{D}(a)} = \exp((\tilde{F}(a) - H_0(X))/\tilde{D}(a)). \tag{17}$$

Here the additional conditions are the following:

$$\int \tilde{f}_0(X, a)\,\mathrm{d}X = \int f(X, a)\,\mathrm{d}X = \int f_0(X)\,\mathrm{d}X = 1,$$
$$\int W(X)\tilde{f}_0(X, a)\,\mathrm{d}X = \int W(X) f(X, a)\,\mathrm{d}X = \langle W\rangle(a).$$

$\langle W\rangle$ is not a constant with respect to the control parameter. Instead of this we trim the mean value of W of the reference state, i.e. of the equilibrium to the actual nonequilibrium level. Hence of interest in this case is only the entropy difference

$$\tilde{S}_0(a) - S(a) = -\int \tilde{f}_0(X, a) \ln \tilde{f}_0(X, a)\,\mathrm{d}X + \int f(X, a) \ln f(X, a)\,\mathrm{d}X.$$

Of course both approaches are equivalent. In this paper we will make use only of the first one, because it is simplier for analytical investigations.

6. Behaviour of the Renormalized Entropy

The solution of (15, 16) is the pair of functions $\tilde{F}(a)$, $\tilde{D}(a)$. It is not possible to find them explicitely, but we can derive expressions for the derivatives $\tilde{F}'(a)$, $\tilde{D}'(a)$. Differentiation of the normalization condition (15) with respect to the control parameter provides

$$\tilde{F}'(a) = -\widetilde{\left\langle \frac{\partial}{\partial a} U \right\rangle} - \tilde{S}(a)\,\tilde{D}'(a). \tag{18}$$

Here and below the average has to be taken over the function (14). We will drop the symbol "$\sim$" over the brackets if on the l.h.s. of the equation stands a renormalized quantity. Differentiating equation (16) we find

$$0 = \int \mathrm{d}X\,\tilde{f}(X, a, W_0)\,W(X)\,\frac{\tilde{F}'(a) + \partial/\partial a\,U(X, a)}{\tilde{D}(a)} - \frac{\tilde{F}(a) - H_0(X) + U(X, a)}{\tilde{D}^2(a)}\,\tilde{D}'(a).$$

Substituting $\tilde{F}'(a)$ from (18) we obtain

$$\tilde{D}'|_{\langle W\rangle}(a) = -\tilde{D}(a)\,\frac{\langle W(X)\,\Delta\,\partial/\partial a\,U\rangle}{\langle W(X)\,\Delta(H_0 - U)\rangle}. \tag{19}$$

(19) can be introduced into (18):

$$\tilde{F}'|_{\langle W\rangle}(a) = -\left\langle \frac{\partial}{\partial a} U \right\rangle + \tilde{S}(a)\,\tilde{D}(a)\,\frac{\langle W\Delta\,\partial/\partial a\,U\rangle}{\langle W\Delta(H_0 - U)\rangle}.$$

With the expressions for $\tilde{D}'(a)$ and $\tilde{F}'(a)$ we find the derivative of the function $\tilde{f}$, cf. (14)

$$\frac{\partial}{\partial a}\tilde{f}(X, a, W_0) = \frac{\tilde{f}(X, a, W_0)}{\tilde{D}(a)} \cdot \left(\Delta \frac{\partial}{\partial a} U - \Delta(H_0 - U)\,\frac{\langle W\Delta\,\partial/\partial a\,U\rangle}{\langle W\Delta(H_0 - U)\rangle}\right). \tag{20}$$

From (20) it is easy to see, that in fact $\frac{\mathrm{d}}{\mathrm{d}a}\langle \tilde{W}\rangle = 0$ for any value of "a". Using formula (20) we now are able to differentiate the entropy $\tilde{S}$

$$\frac{\mathrm{d}}{\mathrm{d}a}\tilde{S}|_{\langle W\rangle = W_0}(a) = -\int (1 + \ln \tilde{f})\,\frac{\partial}{\partial a}\tilde{f}\,\mathrm{d}X = \frac{1}{\tilde{D}^2(a)}\left\{\left\langle (H_0 - U)\,\Delta\,\frac{\partial}{\partial a} U\right\rangle - \sigma^2_{(H_0 - U)}\,\frac{\langle W\Delta\,\partial/\partial a\,U\rangle}{\langle W\Delta(H_0 - U)\rangle}\right\}. \tag{21}$$

Here we denoted

$$\sigma^2_{(H_0 - U)} = \sigma^2_{H_0} + \sigma^2_U - 2\langle \Delta H_0\,\Delta U\rangle.$$

$\tilde{S}$ depends on the control parameter and the noise intensity:

$$\tilde{S}|_{\langle W\rangle = W_0} = \tilde{S}(a, \tilde{D}(a), W_0)$$

therefore

$$\frac{\mathrm{d}}{\mathrm{d}a}\tilde{S}|_{\langle W\rangle = W_0} = \left.\frac{\partial \tilde{S}}{\partial a}\right|_{\tilde{D}} + \left.\frac{\partial \tilde{S}}{\partial \tilde{D}}\right|_a \cdot \left.\frac{\partial \tilde{D}}{\partial a}\right|_{\langle W\rangle = W_0}. \tag{22}$$

Comparing (21) and (22) we easily obtain, that the first term on the r.h.s. of (21) is the S-information derivative (cf. (8)), where $\tilde{D}(a)$ has to be substituted for D_0. The second term in (21) is also immediately identified. Indeed, taking into account (19) we get

$$\left.\frac{\partial \tilde{S}}{\partial \tilde{D}}\right|_a = \frac{1}{\tilde{D}^3(a)}\,\sigma^2_{(H_0 - U)}.$$

We now shall study in more detail the case $W(X) = H_0(X)$. First we have to replace (19) by (23)

$$\tilde{D}'|_{\langle H_0\rangle}(a) = -\tilde{D}(a)\,\frac{\langle H_0(X)\,\Delta\,\partial/\partial a\,U(X, a)\rangle}{\langle H_0(X)\,\Delta(H_0(X) - U(X, a))\rangle}. \tag{23}$$

As we have mentioned, we are interested only in real and positive functions $\tilde{D}(a)$. That means that in some situations the scope of this renormalization method in the control parameter space will be limited. If, for example $\frac{\mathrm{d}}{\mathrm{d}a}\langle H_0\rangle > 0$, the denominator of (23), being positive near equilibrium ($|a| \ll 1$) can change its sign at some value $a_0(D_0)$ of the parameter. If we however increase the external noise intensity D_0, $|a_0|$ is increasing too.

The question of the solubility of (23) needs further investigation. In this paper we assume that we always can increase the noise intensity and therefore $\langle H_0 \Delta (H_0 - U)\rangle > 0$ in the interesting parameter range. Note, that the second renormalization method works for arbitrary values $D_0 \left(\text{if } \frac{d}{da} \langle H_0 \rangle > 0\right)$.

Let us write (23) in another form:

$$\frac{d}{da}\frac{1}{\tilde{D}} = \frac{d}{da}\langle H_0\rangle|_{D_0 \to \tilde{D}} \cdot \langle H_0 \Delta (\widetilde{H_0 - U})\rangle^{-1}. \tag{24}$$

For small absolute values of "a" $\tilde{f} \approx f$, therefore

$$\tilde{D}'(a) \approx -\frac{\tilde{D}}{\sigma_{H_0}^2}\widetilde{\left\langle H_0 \Delta \frac{\partial}{\partial a} U\right\rangle} \approx -\frac{D_0^2}{\sigma_{H_0}^2}\frac{d}{da}\langle H_0\rangle.$$

We see that near equilibrium the derivatives of $\tilde{D}$ and $\langle H_0\rangle$ have opposite signs. Moreover, using the idea of the "local equilibrium" [15] this fact can be generalized for all values of the control parameter. If one would use the second renormalization method (cf. Sect. 5) he could obtain the same sign for $\tilde{D}'(a)$ and for $\frac{d}{da}\langle H_0\rangle$ for arbitrary values of a.

Now we write down the entropy derivative, substituting in (21) $H_0(X)$ for $W(X)$

$$\frac{d}{da}\tilde{S}|_{\langle H_0\rangle}(a) = -\frac{1}{\tilde{D}^2(a)}\left\{\left\langle U \Delta \frac{\partial}{\partial a} U\right\rangle - \left\langle H_0 \Delta \frac{\partial}{\partial a} U\right\rangle \frac{\langle (H_0 - U)\Delta U\rangle}{\langle (H_0 - U)\Delta H_0\rangle}\right\}. \tag{25}$$

In the special case (3) we find the simplier result

$$\frac{d}{da}\tilde{S}|_{\langle H_0\rangle}(a) = -\frac{g(a)\,g'(a)\,\sigma_v^2\,\sigma_{H_0}^2}{\tilde{D}^2(a)}\,\frac{1 - r_{H_0,v}^2}{\langle (H_0 - U)\Delta H_0\rangle} \tag{26}$$

$r_{H_0,v}$ is the normalized correlation coefficient:

$$r_{H_0,v} = \frac{\langle \Delta H_0 \Delta v\rangle}{\sigma_{H_0}\sigma_v}, \qquad |r_{H_0,v}| \leqq 1.$$

As we have underlined the denominator in (26) is positive, hence the entropy derivative is negative if $\frac{d}{da} g^2(a) > 0$.

So we can summarize the main results of the one parameter case: Both the entropy S_1 and the renormalized entropy $\tilde{S}$ decrease monotonously with increasing control parameter for one class of systems:

$$\frac{d}{da}S_1(a) \leqq 0 \quad \text{and if } \frac{d}{da}\tilde{S}|_{\langle H_0\rangle}(a) \leqq 0;$$
$$U(X,a) = g(a)\,v(X);$$
$$\frac{d}{da}g^2(a) \geqq 0 \qquad (\text{“}=\text{” only for } a=0). \tag{27}$$

Finishing this section we give the results for the entropy derivatives for Haken's example (9):

$$\frac{d}{d\alpha}S_1 = -a_p\,\sigma_{q^2}^2 \leqq 0,$$

$$\frac{d}{d\alpha}\tilde{S}|_{\langle H_0\rangle} = -\frac{a_p}{\tilde{D}^2}\,\frac{\beta^2\,\sigma_{q^2}^2\,\sigma_{q^4}^2(1 - r_{q^2,q^4}^2)}{-\gamma\alpha\sigma_{q^2}^2 + \beta^2\sigma_{q^4}^2 + \beta(\gamma - \alpha)\cdot\langle \Delta q^2 \Delta q^4\rangle} \leqq 0.$$

7. The Case of Several Control Parameters

All results remain valid here too, we only have to substitute $a \to \mathbf{a} = (a_1, \ldots, a_n)$, $\frac{d}{da} \to \frac{\partial}{\partial a_i}$. For instance we find for the S-information, cf. (8)

$$\frac{\partial}{\partial a_i}S(\mathbf{a}) = -\frac{1}{D_0^2}\left\langle U(X,\mathbf{a})\,\Delta \frac{\partial}{\partial a_i} U(X,\mathbf{a})\right\rangle + \frac{1}{D_0^2}\left\langle H_0(X)\,\Delta \frac{\partial}{\partial a_i} U(X,\mathbf{a})\right\rangle.$$

In the validity of condition (3) we obtain for the firm term:

$$\frac{\partial}{\partial a_i}S_1(\mathbf{a}) = -\frac{1}{D_0^2}\,g_i(a_i)\,g_i'(a_i)\,\sigma_{v_i}^2 - \frac{g_i'(a_i)}{D_0^2}\sum_{j \neq i} g_j(a_j)\langle \Delta v_i \Delta v_j\rangle. \tag{28}$$

In the case of two or more control parameters there do not exist such general conditions for the negativity of the derivative of S_1 like for a single parameter (27), because the signs of the correlators $\langle \Delta v_i \Delta v_j\rangle$ depend on the concrete system.

If, for example, both v_i and v_j are monotonous functions of one single and positive variable E we find (see Appendix)

$$\operatorname{sign}(\langle \Delta v_i \Delta v_j\rangle) = \operatorname{sign}(v_i'(E)\cdot v_j'(E)), \qquad 0 \leqq E \leqq \infty. \tag{29}$$

The expressions for the renormalized quantities remain valid too. So we can write for the noise intensity

$$\frac{\partial}{\partial a_i} \tilde{D}(a)|_{\langle H_0\rangle} = -\tilde{D}(a) \frac{\langle H_0 \Delta \partial/\partial a_i U\rangle}{\langle H_0 \Delta (H_0 - U)\rangle}. \tag{30}$$

For large enough D_0 the denominator is positive again. It is interesting to study the location of the lines $\langle H_0\rangle = \text{const}$ and $\tilde{D}(\mathbf{a})|_{\langle H_0\rangle} = \text{const}$. Along the line $\langle H_0\rangle = \langle H_0\rangle(0)$ the noise intensity remains on the initial level $\tilde{D} = D_0 = \text{const}$. With increasing distance from this line the angle between both lines $\langle H_0\rangle = \text{const}$ and $\tilde{D} = \text{const}$, in general, increases.

The formula for the entropy derivatives $\frac{\partial}{\partial a_i} \tilde{S}(\mathbf{a})$ is the same as (25).

8. Selfsustained Oscillations in a System with two Control Parameters

As an example we study a Van der Pol oscillator with two feedbacks. Computer results had been presented in another paper [18]. Here we like to give the corresponding analytical results, demonstrating the advantage of the general formulas derived above

$$\frac{dx}{dt} = v,$$

$$\frac{d}{dt} v + (a + \tfrac{2}{3} b v^2 + \tfrac{4}{5} c v^4) v + \omega_0^2 x = \sqrt{D}\, y(t),$$

$$\langle y(t)\rangle = 0, \qquad \langle y(t), y(t')\rangle = 2\delta(t-t'),$$

$$D = D_0 = \text{const} \tag{31}$$

$a = \gamma - a_p$, $b = b_0 - b_p$, γ, b_0, a_p, b_p, $c \geqq 0$. The control parameters (feedbacks) are a_p and b_p. Focussing on quasiharmonic oscillations we can derive a Fokker-Planck equation with a stationary solution of the form (1, 2):

$$f(E) = \exp((F - H_0 + U)/D_0),$$

$$H_0(E) = \gamma E + \frac{b_0}{2} E^2 + \frac{c}{3} E^3,$$

$$U(E, a_p, b_p) = a_p E + \frac{b_p}{2} E^2,$$

$$E = (\omega_0^2 x^2 + v^2)/2, \qquad m = 1.$$

With the help of formula (6) we obtain the derivatives of the mean effective energy, which are positive because the condition (7) holds.

$$\frac{\partial}{\partial a_p} \langle H_0\rangle = \frac{1}{D_0}\left(\gamma \sigma_E^2 + \frac{b_0}{2} \langle \Delta E\, \Delta E^2\rangle + \frac{c}{3} \langle \Delta E\, \Delta E^3\rangle\right) \geqq 0,$$

$$\frac{\partial}{\partial b_p/2} \langle H_0\rangle = \frac{1}{D_0}\left(\gamma \langle \Delta E\, \Delta E^2\rangle + \frac{b_0}{2} \sigma_{E^2}^2 + \frac{c}{3} \langle \Delta E^2\, \Delta E^3\rangle\right) \geqq 0. \tag{32}$$

From (13*) we find the derivatives of S_1, which are negative (cf. (7))

$$\frac{\partial}{\partial a_p} S_1 = -\frac{1}{D_0^2} a_p \sigma_E^2 - \frac{1}{D_0^2} \frac{b_p}{2} \langle \Delta E\, \Delta E^2\rangle \leqq 0,$$

$$\frac{\partial}{\partial b_p/2} S_1 = -\frac{1}{D_0^2} \frac{b_p}{2} p \sigma_{E^2}^2 - \frac{1}{D_0^2} a_p \langle \Delta E\, \Delta E^2\rangle \leqq 0. \tag{33}$$

From (32) and (33) we obtain for the S-information (cf. (12))

$$\frac{\partial}{\partial a_p} S = \frac{1}{D_0^2}\left(a \sigma_E^2 + \frac{b}{2} \langle \Delta E\, \Delta E^2\rangle + \frac{c}{3} \langle \Delta E\, \Delta E^3\rangle\right),$$

$$\frac{\partial}{\partial b_p/2} S = \frac{1}{D_0^2}\left(a \langle \Delta E\, \Delta E^2\rangle + \frac{b}{2} \sigma_{E^2}^2 + \frac{c}{3} \langle \Delta E^2\, \Delta E^3\rangle\right).$$

The partial derivatives of the S-information can change their sign. In dependence on the relation between a, b and c the S-information is a monotonously increasing function of the pumpings or exhibits one maximum.

Finally, we obtain the expressions for the renormalized entropy (26). Because they are rather complicated we confine here to the case $c = 0$.

$$\frac{\partial}{\partial a_p} \tilde{S} = -\frac{1}{\tilde{D}_0^2} \frac{b}{2} \sigma_E^2 \sigma_{E^2}^2 \cdot \frac{(1 - r_{E,E^2}^2)(a_p b_0/2 - \gamma b_p/2)}{\gamma a \sigma_E^2 + b_0/2\, b/2\, \sigma_{E^2}^2 + (\gamma b/2 + a b_0/2)\langle \Delta E\, \Delta E^2\rangle},$$

$$\frac{\partial}{\partial b_p/2} \tilde{S} = \frac{1}{\tilde{D}_0^2} a \sigma_E^2 \sigma_{E^2}^2 \cdot \frac{(1 - r_{E,E^2}^2)(a_p b_0/2 - \gamma b_p/2)}{\gamma a \sigma_E^2 + b_0/2\, b/2\, \sigma_{E^2}^2 + (\gamma b/2 + a b_0/2)\langle \Delta E\, \Delta E^2\rangle}. \tag{34}$$

Within the scope of the renormalized method the denominator is positive.

It is interesting to study the gradient of entropy in the control parameter plane:

$$\mathbf{grad}\,\tilde{S}=\left\{\frac{\partial}{\partial a_p}\tilde{S};\ \frac{\partial}{\partial b_p/2}\tilde{S}\right\}.$$

For the angle between the gradient and the axis a_p in an arbitrary point we can write

$$\tan\alpha\left(a_p,\frac{b_p}{2}\right)=\frac{\partial}{\partial b_p/2}\tilde{S}\Big/\frac{\partial}{\partial a_p}\tilde{S}=-2\,\frac{\gamma-a_p}{b_0-b_p}.$$

This formula describes a family of concentric circles with the center $(\gamma, b_0/2)$. The equientropy curvatores are straight lines starting from this point. The line of maximum entropy is $b_p=b_0\,a_p/\gamma$, along it the partial derivatives vanish (34). In the point $(\gamma, b_0/2)$ the s.p.d.f. is not normalizable, this state cannot be reached. Indeed, in the effective hamiltonian are always higher nonlinearities than in $U(X,a)$, e.g. $c\neq 0$. Therefore in real systems one will not find such extraordinary points. If $c\neq 0$ both the partial derivatives of entropy are negative. Thus, for arbitrary positive "c" there exist ranges in which the second control parameter allows us to diminish the entropy further. So we come to a complete agreement with the computer results [18].

9. Weakly Nonlinear Systems

In some papers [3, 12, 20–22] the entropy for a system given by a Langevin equation was calculated with fixing $\langle E\rangle$ instead of $\langle H_0\rangle$. In the case of oscillators E is the oscillation energy (cf. Sect. 8). Of course the results for the entropy derivatives change. This can be seen from the general formula (21).

Let us divide the effective hamiltonian into a linear and a nonlinear part: $H_0(E)=\gamma E+H_0^{nl}(E)$. If the second term is small, naturally $\langle H_0\rangle\approx\langle E\rangle$.

The entropy derivatives for the oscillator with two pumpings, studied in Sect. 8 had been derived in works of Ebeling, Engel-Herbert and co-workers [20–22]. This results can be easily found making use of the general formula (21), where E has to be substituted for W. We provide them for sake of completness:

$$\frac{\partial}{\partial a_p}\tilde{S}|_{\langle E\rangle}=-\frac{1}{\tilde{D}^2}\,\sigma_E^2\,\sigma_W^2\cdot\frac{1-r_{E,W}^2}{a\sigma_E^2+b/2\langle\Delta E\,\Delta E^2\rangle+c/3\langle\Delta E\,\Delta E^3\rangle},$$

$$\frac{\partial}{\partial b_p/2}\tilde{S}|_{\langle E\rangle}=\frac{\sigma_E\,\sigma_{E^2}\,\sigma_y\,\sigma_z}{\tilde{D}^2}\cdot\frac{r_{E,y}\cdot r_{E^2,z}-r_{E,E^2}\cdot r_{y,z}}{a\sigma_E^2+b/2\langle\Delta E\,\Delta E^2\rangle+c/3\langle\Delta E\,\Delta E^3\rangle}$$

where we denoted $W=H_0^{nl}-\frac{b_p}{2}E^2$, $y=aE+\frac{c}{3}E^3$, $z=\frac{b}{2}E^2+\frac{c}{3}E^3$. For the case $b_p=c=0$ it was shown explicitely, that the denominator $\left(\text{it is } D_0^2*\frac{\partial}{\partial a_p}S\text{, cf. (8)}\right)$ is positive.

10. Discussion

As we have shown the information entropy (S-information) is in general a nonmonotonous function of the distance from equilibrium in open systems, because it depends on the mean energy. There may be various approaches to calculate the entropy of states at *constant* values of the mean energy.

One method is a renormalization of temperature or intensity of external noise respectively. Of course, this leads to a deformation of the s.p.d.f. The question is, in which situations this deformation is relevant and in which not. For a special class of functions (1) changes of temperature or noise respectively do not change the features of the system qualitatively, particular the location and the type of the extrema of the s.p.d.f., i.e. the location and stability of stationary states, and so on.

If we increase the noise (provided that $\langle H_0\rangle(0)<\langle H_0\rangle(a)$) in the equilibrium distribution function (second way), its maximum remains on the original position. The height of the maximum decreases but the function is broadening. Of course, this procedure can be continued. However if we use the first way we have to diminish the noise intensity in the non-equilibrium p.d.f. That means that all maxima will rise in hight but squeeze more and more. Indeed this procedure may break down if the renormalized noise vanishes. In the opposite case ($\langle H_0\rangle(0)>\langle H_0\rangle(a)$) it is the first renormalization method that can be used for all parameter values.

The situation changes if we study stationary probability distribution functions with a noise dependent effective hamiltonian, e.g. multiplicative noise. In this case the features of the state (see above) depend on the noise intensity too. Here we have to restrict ourselves to a local approach [15]: We consider only a small surroundings of the state we are interested in, so that within this range the renormalized noise intensity does not change very much.

As we have seen for the renormalization method the choice of the quantity to be fixed is important. We think that the effective hamiltonian (2) should be preferred because it coincides with the Hamilton function if we deal with a thermostat-system [18].

Let us summarize the main properties of the entropy calculated at $\langle H_0\rangle=\text{const}$:

1. $\tilde{S}|_{\langle H_0\rangle}$ is maximal in equilibrium, i.e. if $U=0$ (Gibbs theorem, cf. [15, 18]).

2. The entropy is decreasing monotonously if the intensity of the pumping is increased (case (3)). In the case of one control parameter it has been shown analytically, cf. (27), for two parameters we base on numerical results.

3. The results depend only on the s.p.d.f. but not on the nature of the particular system.

The entropy $\tilde{S}|_{\langle H_0\rangle}$ can be used in a constructive way [11, 14, 15, 18]: The decrease of entropy can serve as a necessary condition for a selforganization process. Moreover, the difference $S(a=0)-\tilde{S}|_{\langle H_0\rangle}(a)$ can be used as a measure of the distance from equilibrium, i.e. of the degree of order.

Appendix. Proof of Condition (8)

Integration of (5) over E from 0 to ∞ gives

$$\int_0^\infty \frac{\partial}{\partial a} f(E,a)\,\mathrm{d}E = \frac{1}{D_0}\int_0^\infty \left(\frac{\partial}{\partial a}U - \left\langle\frac{\partial}{\partial a}U\right\rangle\right) f(E,a)\,\mathrm{d}E = 0. \qquad \text{(I)}$$

We presume that there exists a value E_1: For all $E > E_1$ $\frac{\partial}{\partial a} f(E,a) \geqq 0$ ($\leqq 0$). E_1 is the maximum solution of the equation $\frac{\partial}{\partial a} U(a,E) = \left\langle \frac{\partial}{\partial a} U\right\rangle$. Then we obtain from (I)

$$\int_0^{E_1} \frac{\partial}{\partial a} f(E,a)\,\mathrm{d}E = -\int_{E_1}^\infty \frac{\partial}{\partial a} f(E,a)\,\mathrm{d}E \leqq 0 \qquad (\geqq 0).$$

Using this result we can write for the derivative of the average of any monotonously increasing function $A(E)$, cf. (7)

$$\frac{\mathrm{d}}{\mathrm{d}a}\langle A\rangle = \frac{1}{D_0}\int_0^{E_1} \mathrm{d}E\,A(E)\,\Delta\frac{\partial}{\partial a}U\cdot f + \frac{1}{D_0}\int_{E_1}^\infty \mathrm{d}E\,A(E)\,\Delta\frac{\partial}{\partial a}U\cdot f. \qquad \text{(II)}$$

For continuous functions $A(E)$, $f(E,a)$ we can write

$$0 \geqq \int_0^{E_1} dE\,A(E)\,\Delta\frac{\partial}{\partial a}U\cdot f \geqq A(E_1)\int_0^{E_1}\mathrm{d}E\,\Delta\frac{\partial}{\partial a}U\cdot f \qquad (\leqq)$$

$$\int_{E_1}^\infty \mathrm{d}E\,A(E)\,\Delta\frac{\partial}{\partial a}U\cdot f \geqq A(E_1)\int_{E_1}^\infty \mathrm{d}E\,\Delta\frac{\partial}{\partial a}U\cdot f \geqq 0 \qquad (\leqq).$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}a}\langle A\rangle \geqq \frac{A(E_1)}{D_0}\int_0^\infty \mathrm{d}E\,\Delta\frac{\partial}{\partial a}U\cdot f = 0 \qquad (\leqq).$$

If $A(E)$ is a monotonously decreasing function the sign of the derivative (II) changes, hence we obtain (7).

This proof remains valid for the following functions too:

$$\frac{\mathrm{d}}{\mathrm{d}E}A(E) \geqq 0 \quad (\leqq) \qquad \text{if } E \geqq E_1$$

and

$$A(E) \leqq A(E_1) \quad (\geqq) \qquad \text{if } E \leqq E_1.$$

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