Kapitel 1

Introduction to canonical quantization

Many physical systems are well described by continuum models, i.e. in terms of fields. Examples are the displacement field $\mathbf{q}(\mathbf{r}, t)$ of particles in a continuous medium such as a gas, a liquid or a plasma. Another example is the electromagnetic field described by the electric and magnetic field strength $\mathbf{E}(\mathbf{r}, t)$, $\mathbf{B}(\mathbf{r}, t)$ which are governed by Maxwell's equations. Furthermore, also particles can be described by fields. In fact, quantum mechanics has changed our picture of the micro-world in such a way that point particles, i.e. concentration of matter in an infinitesimally small space point is meaningless. Instead, matter is de-localized in space which is described by probability densities. These again are field-like continuous functions which again obey equations of field theory. Depending on the particle type, this gives rise to the "Schrödinger field" which is associated with non-relativistic particles or with the "Klein-Gordon field" and the "Dirac field", in the case of relativistic particles.

1.1 Lagrange functional and minimal action principle

A very elegant way to derive equations of motion of physical systems is the minimal action principle. We are now going to generalize this principle to arbitrary fields and derive general equations of motion – the Euler-Lagrange equations for the Lagrange functional. A remarkable property of this approach is that it yields, in a straightforward and general way, the basic conservation laws of any physical theory and their relation to the intrinsic symmetries of the system.

Let the state of a general continuous system be completely described by a finite number M of fields,

$$\Phi(\mathbf{r},t) = \{\Phi_1(\mathbf{r},t), \dots \Phi_M(\mathbf{r},t)\}, \tag{1.1}$$

which can be considered as independent generalized "variables" of the system. The Φ_k can be scalar or vector fields, real or complex and are defined in the volume V and time interval $t_i \le t \le t_f$ which form a four-dimensional space time element $\Delta\Omega$. In fact, the theory should be Lorentz invariant, i.e. symmetric with respect to space and time variables. We can make this symmetry more explicit by introducing the four vector notation for coordinates and derivatives (summation over identical subscript-superscript index pairs is implied)

$$\begin{aligned} x^\mu &= \left(x^0, \vec{x}\right) = (ct, \mathbf{r})\,, \quad \mu = 0,1,2,3, && (1.2)\\ x_\mu &= g_{\mu\nu} x^\nu = \left(x^0, -\vec{x}\right), && (1.3) \end{aligned}$$

where $g^{\mu\nu}$ is the metric tensor

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{1.4}$$

and a scalar product of two 4−vectors is given by $a_\mu b^\mu = a^\mu b_\mu = a_0 b_0 - \vec{a}\vec{b}$ and is a Lorentz scalar (Lorentz invariant). We will also need the metric tensor with mixed indices,

$$g^\mu_\nu = g^\nu_\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \delta_{\mu,\nu}. \tag{1.5}$$

Finally, we also note the way how to transform two-component tensors with various superscript and subscript properties,

$$\begin{aligned} F^{\mu\nu} &= g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta}, && (1.6)\\ F_{\mu\nu} &= g_{\mu\alpha} g_{\nu\beta} F^{\alpha\beta}, && (1.7)\\ F^\nu_\mu &= g_{\mu\alpha} F^{\alpha\nu}, && (1.8) \end{aligned}$$

which is straightforwardly generalized to more complicated tensor quantities.

Using the 4-vectors we can construct two four-dimensional differential operators according to

$$\partial^\mu = \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial ct}, \vec{\nabla}\right), \tag{1.9}$$

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial ct}, -\vec{\nabla}\right), \tag{1.10}$$

giving rise to a four-dimensional divergence or gradient which will be used frequently. For example, the 4-divergence of a 4-vector B^μ is a Lorentz skalar

$$\partial_\mu B^\mu = \frac{\partial}{\partial x^\mu} B^\mu = \frac{\partial}{\partial ct} B^0 + \vec{\nabla}\mathbf{B} = \text{inv}, \tag{1.11}$$

as is the 4-dimensional generalization of the Laplace operator, the D'Alambert operator

$$\partial_\mu \partial^\mu = \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \vec{\nabla}(-\vec{\nabla}) \equiv \Box = \text{inv}. \tag{1.12}$$

Note the sign change in front of the spatial part in Eq. (1.12) which does not occur in Eq. (1.11).

The physical properties of the system will then be determined by the fields Φ and their time and space derivatives, $\dot{\Phi}, \ddot{\Phi}, \ldots$ and $\Phi', \Phi'', \ldots$. There is no a-priori rule how the dependence of the Lagrange density on these functions should look like. This dependence could be non-local, i.e. $\mathcal{L}(\mathbf{r}, t)$ could depend on the fields at different space time points, e.g. contain terms of the form $\int d\bar{t} \int d^3\bar{r}\, K(\mathbf{r}, t; \bar{\mathbf{r}}, \bar{t})\Phi(\bar{\mathbf{r}}, \bar{t})$ or second or higher powers in the fields, products or other nonlinear combinations. Furthermore, $\mathcal{L}$ could, in principle, depend on space and time explicitly, $\mathcal{L} = \mathcal{L}(\mathbf{r}, t; \Phi, ...)$. An explicit time dependence would be reasonable only if the system undergoes a time evolution which is controlled from the outside and not an intrinsic process described by the dynamics of the fields. Similarly, an explicit space dependence could be expected if the system is subject to an external potential which leads to different physics in different space points. If this is not the case an explicit r and t dependence may be ruled out. Finally, the fundamental requirement to a theory of physical processes should be its intrinsic simplicity. Therefore, we will start to construct the Lagrange density using only local expressions and only linear dependencies on all fields and their space and time derivatives[1].

[1]Interestingly, all experimentally verified theories which are known so far can be derived from Lagrangians of this simple structure.

Thus we choose the following structure of the Lagrange density as an ansatz:

$$\mathcal{L}(\mathbf{r},t) = \mathcal{L}[\Phi(\mathbf{r},t), \dot{\Phi}(\mathbf{r},t), \Phi'(\mathbf{r},t)] = \mathcal{L}[\Phi(\mathbf{r},t), \Phi_\mu(\mathbf{r},t)], \tag{1.13}$$

where everywhere Φ is understood as a vector with M components (1.1) and Φ_μ denotes the four-dimensional derivative vector (gradient), $\partial_\mu = (\partial_t\Phi, -\nabla\Phi)$. Using $\mathcal{L}$ we find a central quantity of theoretical physics – the action – defined by

$$\boxed{S = \int dt \int d^3r\, \mathcal{L} = \int_\Omega d^4x\, \mathcal{L}(\mathbf{r},t)} \tag{1.14}$$

The Lagrange function gives the possibility to derive the physical equations of motion in a very general way based on the *Minimal action principle*: Those fields Φ which obey the physical equations of motion and fulfill boundary conditions at two space-time points 1 and 2 minimize the action S. We will call these particular realizations of the fields Φ_{phys}. This principle can be turned around: Those fields which minimize the action and fulfill the boundary conditions are the "true" physical fields Φ_{phys}. This latter formulation shows the power and beauty of this principle: it provides a completely general approach to physical systems, independently of the particular area and specific kind of fields involved.

Let us now find the extremum of the action [2]. To this end we compute the variation of S around the physical fields, i.e. in the "point" $\Phi = \Phi_{phys}$, and put it equal to zero,

$$0 = \delta S = \int_1^2 d^4x \left\{ \frac{\delta\mathcal{L}}{\delta\Phi}\delta\Phi + \frac{\delta\mathcal{L}}{\delta\Phi_\mu}\delta\Phi_\mu \right\}. \tag{1.15}$$

Assuming that the fields Φ are continuous functions we can change the order of differentiation and variation, i.e. $\delta\Phi_\mu = \partial_\mu\delta\Phi$. In order to express the variation $\partial_\mu\delta\Phi$ by $\delta\Phi$ we perform a partial integration of the second term in Eq. (1.15) neglecting the terms at the boundaries of the integral by making use of the boundary condition $\delta\Phi(1) = \delta\Phi(2) = 0$. As a result we obtain

$$0 = \int_1^2 d^4x \left\{ \frac{\delta\mathcal{L}}{\delta\Phi} - \partial_\mu \frac{\delta\mathcal{L}}{\delta\Phi_\mu} \right\} \delta\Phi. \tag{1.16}$$

Since this equation should be fulfilled for arbitrary fluctuations $\delta\Phi$ we have to

[2] In practically all cases, finding the extremum is sufficient. See also problem 1.

require that the term in the parantheses vanishes

$$
\begin{aligned}
0 &= \frac{\delta\mathcal{L}}{\delta\Phi} - \partial_\mu \frac{\delta\mathcal{L}}{\delta\Phi_\mu} = \\
&= \frac{\delta\mathcal{L}}{\delta\Phi} - \frac{\partial}{\partial t}\frac{\delta\mathcal{L}}{\delta\dot{\Phi}} - \vec{\nabla}\frac{\delta\mathcal{L}}{\delta\vec{\nabla}\Phi} .
\end{aligned}
\tag{1.17}
$$

These are the famous Euler-Lagrange equations, the general equations of motion of the field(s) Φ. Note that the integral (1.16) has to be understood as a scalar product of two $M-$dimensional vectors. Since the fluctuations of the individual fields $\delta\Phi_1, \dots \delta\Phi_M$ are independent (since the fields are assumed to be independent variables), vanishing of δS requires that all M terms in the parantheses vanish simultaneously, so (1.18) is equivalent to

$$
\begin{aligned}
0 &= \frac{\delta\mathcal{L}}{\delta\Phi_k} - \partial_\mu \frac{\delta\mathcal{L}}{\delta\Phi_{k,\mu}} = \\
&= \frac{\delta\mathcal{L}}{\delta\Phi_k} - \frac{\partial}{\partial t}\frac{\delta\mathcal{L}}{\delta\dot{\Phi}_k} - \vec{\nabla}\frac{\delta\mathcal{L}}{\delta\vec{\nabla}\Phi_k}, \qquad k = 1, \dots M.
\end{aligned}
\tag{1.18}
$$

Let us briefly discuss this result.

1. The variational principle is of course a postulate which cannot be proven, similar to Newton's equations, Maxwell's equations or the Schrödinger equation. Vice versa, assuming the validity of one of the latter equations one can show the existence of a Lagrangian which obeys the minimal action principle for this particular system. However, starting from the minimal action principle we have a universal principle method yielding all these equations at once.

2. When accepting the validity of the minimal action principle, the main tasks of physical theories consist in deriving (or guessing) explicit expressions for the Lagrange density.

3. A remarkable property of the variaional principle is that it yields local (space and time-dependent) solutions $\Phi_{phys}(\mathbf{r}, t)$ from minimizing a single global scalar function – the action which is an integral over a space-time volume Ω.

4. The Euler-Lagrange equations (ELE) are Lorentz invariant as they contain only Lorentz scalars which is seen from the first equation (1.18).

5. The ELE do not depend on the choice of the boundary of the volume Ω. In fact, we expect the physical equations of motion to be independent of

time and space. On the other hand, if space or time are inhomogeneous, i.e. the Lagrangian explicitly depends on r or t we cannot exclude also a dependence on the chosen boundary points 1 and 2.

6. When deriving the ELE we implicitly have assumed that the Lagrange density is a sufficiently smooth functional of the fields and their time and space derivatives and also that the time derivative and the gradient of all fields are continuous.

7. When finding the extremum of the action given by the ELE we cannot rule out that this result corresponds to a maximum of S. Strictly speaking, we have to verify that the second variation of the action is negative (see problem 1).

1.1.1 Classical mechanics of a point particle

Before proceeding we consider the simplest case of a point particle of mass m in a $1D$ external potential $U(x)$. Then, the field is replaced by the coordinate, $\Phi(\mathbf{r}, t) \to q(t)$ and $\mathcal{L} \to L$ where the Lagrangian is given by kinetic (T) and potential energy (V) according to

$$L(q, \dot{q}) = T - V = \frac{m}{2}\dot{q}^2 - U(q). \tag{1.19}$$

Then the Euler-Lagrange equations (1.18) become

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0, \tag{1.20}$$

yielding the equation of motion for the "physical" variable $q(t)$

$$m\ddot{q} = -\frac{\partial U}{\partial q}, \tag{1.21}$$

i.e. we recover Newton's equation. Knowing the Lagrange function, mechanics tells us how to obtain from it the momentum p and the hamilton function $H(q, p)$,

$$p = \frac{\partial L}{\partial \dot{q}} = m\dot{q}, \tag{1.22}$$

$$H(q, p) = p\dot{q}(p) - L[q, \dot{q}(p)] = \frac{p^2}{2m} + U(q). \tag{1.23}$$

Using the hamitonian we obtain an alternative form of the equations of motion: two first-order equations (Hamilton's equations) for q and p,

$$\dot{p} \;=\; -\frac{\partial H}{\partial q} = -\frac{\partial U}{\partial q}, \tag{1.24}$$

$$\dot{q} \;=\; \frac{\partial H}{\partial p} = \frac{p}{m}. \tag{1.25}$$

Finally, we recall another result of point mechanics, now extending the analysis to N particles: any function F depending only on the canonical variables $q_1 \dots q_N, p_1 \dots p_N$ has a simple time evolution given by the Poisson bracket with the Hamiltonian,

$$\dot{F} \;=\; \{F, H\}\,, \tag{1.26}$$

$$\{F, H\} \;=\; \sum_{i=1}^{N} \left(\frac{\partial F}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q_i} \right). \tag{1.27}$$

1.1.2 Canonical momentum and Hamilton density of classical fields

We now return to the general problem described by the fields $\Phi_1 \dots \Phi_M$. The example of the point particle suggests to define a "momentum" $\pi_l(\mathbf{r}, t)$, i.e. the field which is "canonically" adjoint to Φ_l, by defining, in analogy to (1.22),

$$\boxed{\pi_l(\mathbf{r}, t) = \frac{\delta \mathcal{L}}{\delta \dot{\Phi}_l(\mathbf{r}, t)} = \pi_l^0(\mathbf{r}, t), \qquad l = 1, \dots M} \tag{1.28}$$

Thus the canonical momentum field follows from the functional derivative of the Lagrange density with respect to the time derivative of the field. As indicated by the last equality, this can also be understood as the 0 component of a 4-vector π^μ defined as

$$\pi_l^\mu(\mathbf{r}, t) \;=\; \frac{\delta \mathcal{L}}{\delta_\mu \Phi_l(\mathbf{r}, t)}, \tag{1.29}$$

but only the zero component of π^μ has the physical meaning of a momentum. Yet the whole 4-vector can be conveniently used to make the expressions more compact. By combining the 4-vectors π_l^μ into a momentum vector of all fields by defining $\pi^\mu = \{\pi_1^\mu, \dots \pi_M^\mu\}$ we can rewrite Eq. (1.18) compactly as

$$\boxed{\frac{\delta \mathcal{L}}{\delta \Phi} - \partial_\mu \pi^\mu = 0} \tag{1.30}$$

Using again the analogy with the point mechanics, we can now introduce a Hamilton density $\mathcal{H}$ which is related to the hamiltonian by $H(t) = \int d^3r\,\mathcal{H}(\mathbf{r},t)$. Generalizing Eq. (1.23) to fields we can write

$$\mathcal{H}(\mathbf{r},t) \;=\; \mathcal{H}[\Phi,\pi] = \pi\dot{\Phi}[\pi] - \mathcal{L}, \tag{1.31}$$

where the time-derivative of Φ has to be eliminated to recover the dependence of $\mathcal{H}$ on the two independent variables Φ and π. Interestingly, also in the case of continuous fields there exist generalized Hamilton's equations which are analogous to those of point mechanics, Eqs. (1.24, 1.25). To verify this, we consider the variation of H, performing the variation of $\mathcal{H}$ and $\mathcal{L}$ using the definition of π and the Euler Lagrange equations (1.18),

$$\begin{aligned}
\delta H &= \int d^3r\,\left\{\pi\delta\dot{\Phi} + \dot{\Phi}\delta\pi - \delta\mathcal{L}\right\} \\
&= \int d^3r\,\left\{\pi\delta\dot{\Phi} + \dot{\Phi}\delta\pi - \frac{\delta\mathcal{L}}{\delta\Phi}\delta\Phi - \frac{\delta\mathcal{L}}{\delta\Phi_\mu}\delta\Phi_\mu\right\} \\
&= \int d^3r\,\left\{\pi\delta\dot{\Phi} + \dot{\Phi}\delta\pi - \frac{\delta\mathcal{L}}{\delta\Phi}\delta\Phi - \pi\delta\dot{\Phi} - \delta\Phi\vec{\nabla}\frac{\delta\mathcal{L}}{\delta\vec{\nabla}\Phi}\right\} \\
&= \int d^3r\,\left\{\dot{\Phi}\delta\pi - \dot{\pi}\delta\Phi\right\} \equiv \int d^3r\,\delta\mathcal{H}[\Phi,\pi].
\end{aligned} \tag{1.32}$$

In the last term of the third line we have performed an integration by parts. The two terms containing derivatives of $\mathcal{L}$ are then replaced by the time derivative of π using Eq. (1.18). Since $\mathcal{H}[\Phi,\pi]$ is a functional of the two independent fields Φ and π, it is evident from the last equation that

$$\dot{\pi} \;=\; -\frac{\delta\mathcal{H}}{\delta\Phi}, \tag{1.33}$$

$$\dot{\Phi} \;=\; \frac{\delta\mathcal{H}}{\delta\pi}, \tag{1.34}$$

i.e. we have obtained another variant of the equations of motion of the canonical field – the continuum generalization of Hamilton's equations. This confirms the consistency of the definitions of the canonical momentum and of the Hamilton density.

Finally, we generalize another result of point mechanics – the time evolution of any function F depending only on the canonical variables q_i, p_i of N particles which is given by the Poisson brackets, cf. Eq. (1.27). To this end we introduce the spatial density f of F and compute the time derivative, using Hamilton's equations (1.33), (1.34)

$$\begin{aligned} \dot{F}(t) &= \int d^3r \left\{ \frac{\delta f}{\delta \Phi} \dot{\Phi} + \frac{\delta f}{\delta \pi} \dot{\pi} \right\} \\ &= \int d^3r \left\{ \frac{\delta f}{\delta \Phi} \frac{\delta \mathcal{H}}{\delta \pi} - \frac{\delta f}{\delta \pi} \frac{\delta \mathcal{H}}{\delta \Phi} \right\} = \{F, H\}, \end{aligned} \tag{1.35}$$

where in the last line we introduced the continuum generalization of the Poisson bracket. Compared to the discrete case, Eq. (1.27), here the sum over the particles is replaced by a space integration.

1.2 Conservation laws in classical field theory

We have seen in the last section that minimizing the action yieds the physical equations of motion.The key quantity in this approach is the Lagrange density. It is, therefore, important to know whether the definition of $\mathcal{L}$ is unique or whether there exists any freedom in choosing it. In fact, the answer is given by a very simple statement:

Theorem: *The equations of motion (1.18) remain invariant under any transformation of $\mathcal{L}$ of the form*

$$\mathcal{L} \longrightarrow \mathcal{L} + \delta\tilde{\mathcal{L}} \tag{1.36}$$

$$\text{with} \quad \delta\tilde{\mathcal{L}}(x) = \partial_\mu W^\mu(x), \tag{1.37}$$

where W is a continuous function of the space and time arguments and vanishes at the boundary $\delta\Omega$ of the four-dimensional volume Ω.

Proof:
Consider the variation of the action and its change under the above transformation

$$\delta S = \int_1^2 d^4x \, \delta\mathcal{L} \longrightarrow \int_1^2 d^4x \, \{\delta\mathcal{L} + \partial_\mu W^\mu(x)\}. \tag{1.38}$$

The additional term can be converted into a surface integral by using the four-dimensional Gauss theorem

$$\int_1^2 d^4x \, \partial_\mu W^\mu(x) = \oint_{\delta\Omega} dS_\mu W^\mu = 0,$$

which is zero since W is assumed to vanish at the boundary and the theorem is proven.

So what does this transformation mean? First, the transformation $\delta\tilde{\mathcal{L}}(x)$ is understood as an infinitesimal transformation so its variation is the function itself. Second, the particular form (1.37) means that the addition to $\mathcal{L}$ is a full 4-divergence, i.e. W has to have the form of a 4-vector $W^\mu = (W^0, \vec{w})$ and $\delta\tilde{\mathcal{L}}(x)$ is a Lorentz scalar of the form

$$\delta\tilde{\mathcal{L}}(x) = \frac{\partial W^0}{\partial t} - \operatorname{div}\vec{w}. \tag{1.39}$$

Thus there is a certain flexibility in defining $\mathcal{L}$ without changing the equations of motion derived from minimzing the action[3]. Now the question is how to exploit this freedom. One way to produce such a transformation of $\mathcal{L}$ is to make a transformation of the fields Φ. We will call a transformation of the fields which generates the change (1.37) of $\mathcal{L}$ a *Symmetry transformation.* This means

$$\begin{aligned} \Phi(x) &\rightarrow \Phi(x) + \delta\Phi(x), \\ \mathcal{L}[\Phi(x)] &\rightarrow \mathcal{L}[\Phi(x) + \delta\Phi(x)] \equiv \mathcal{L}[\Phi(x)] + \partial_\mu W^\mu[\delta\Phi(x)]. \end{aligned} \tag{1.40}$$

Furthermore we require that the transformation is continuous and exists for all fields Φ, not just the physical fields Φ_{phys} which obey the Euler Lagrange equations. Under these conditions there exists a generalized current density j^μ which obeys

Noether's Theorem: [4] *For any continuous symmetry transformation $\delta\Phi$ of the form (1.40) there exists a 4-current density of the form (1.41). For all physical fields Φ_{phys} this current density has a vanishing 4-divergence, i.e.*

$$j^\mu(x) = \pi^\mu(x)\delta\Phi(x) - W^\mu(x), \tag{1.41}$$

$$\partial_\mu j^\mu(x) = 0, \tag{1.42}$$

where π^μ is defined in (1.29).

Proof:

The 4-divergence of the change of the Lagrange density associated with the

[3]This situation is similar to gauge invariance in electrodynamics where different forms of the electromagnetic potentials may be chosen without changing the physical observables (the electromagnetic field and the quantum-mechanical probability density)

[4]The theorem is due to Emmy Noether who formulated it in 1918.

transformation $\delta\Phi$ is the total variation of $\mathcal{L}$

$$\begin{aligned} \partial_\mu W^\mu &= \delta\mathcal{L} = \frac{\delta\mathcal{L}}{\delta\Phi}\delta\Phi + \pi^\mu\delta\Phi_\mu \\ &= \left\{\frac{\delta\mathcal{L}}{\delta\Phi} - \partial_\mu\pi^\mu\right\}\delta\Phi + \partial_\mu\left(\pi^\mu\delta\Phi\right). \end{aligned} \tag{1.43}$$

Here we used $\delta\Phi_\mu = \partial_\mu\delta\Phi$, and the derivative of π^μ in the last term on the r.h.s. is compensated by the second term in the parantheses. So far these were identical transformations valid for arbitrary fields Φ. Now we specialize to the physical fields which obey the Euler Lagrange equations (1.18) which just appear in the parantheses, i.e. the first term on the r.h.s. of (1.43) vanishes for $\Phi = \Phi_{phys}$, and we may rewrite (1.43)

$$0 = \partial_\mu\left(\pi^\mu\delta\Phi - W^\mu\right) = \partial_\mu j^\mu.$$

Thus we have confirmed the vanishing of the 4-divergence of the 4-current density and obtained the explicit form of this density which exactly agrees with the statement of the theorem.

1.2.1 Translational invariance. Energy and momentum conservation

We now consider the simplest symmetry transformation – an infinitesimal space time transformation

$$\begin{aligned} x^\mu &\longrightarrow x^\mu + a^\mu \\ \Phi\left(x^\mu\right) &\longrightarrow \Phi\left(x^\mu + a^\mu\right). \end{aligned} \tag{1.44}$$

The corresponding symmetry transformation $\delta\Phi$ is then obtained by Taylor expanding the fields in the shifted arguments around the original value

$$\delta\Phi = \Phi\left(x^\mu + a^\mu\right) - \Phi\left(x^\mu\right) = a_\mu\partial^\mu\Phi(x) + \dots, \tag{1.45}$$

where second order and higher terms are neglected. This is a superposition of four orthogonal translations – one in time and three in space. Owing to independence of these translations we can consider (any) one of them choosing $\mu = \alpha$, where $\alpha = 0, \dots 3$. Further, we may rescale the coordinate system such that the shift $a_\alpha = 1$. Then the symmetry transform and its derivative are simply

$$\delta\Phi \approx \partial^\alpha\Phi(x), \tag{1.46}$$

$$\delta\Phi_\mu \approx \partial^\alpha\Phi_\mu(x). \tag{1.47}$$

Let us now compute the Noether current associated with translations, i.e. with the symmetry transfrom (1.46). The variation of the Lagrange density $\mathcal{L}$ will now be equivalent to space-time variation of its arguments Φ and Φ_μ, i.e.

$$\begin{aligned}\delta\mathcal{L}(x) &= \frac{\delta\mathcal{L}}{\delta\Phi}\delta\Phi + \frac{\delta\mathcal{L}}{\delta\Phi_\mu}\delta\Phi_\mu = \\ &= \frac{\delta\mathcal{L}}{\delta\Phi}\partial^\alpha\Phi + \frac{\delta\mathcal{L}}{\delta\Phi_\mu}\partial^\alpha\Phi_\mu \equiv \partial^\alpha\mathcal{L}(x).\end{aligned} \tag{1.48}$$

Since for a symmetry transformation, the variation of $\mathcal{L}$ can be written as a 4−divergence, cf. Eq. (1.40), we can rewrite [transforming to a lower derivative using Eq. (1.3)]

$$\partial_\mu W^\mu \equiv \delta\mathcal{L}(x) = \partial^\alpha\mathcal{L}(x) = \partial_\mu\left[g^{\mu\alpha}\mathcal{L}(x)\right]. \tag{1.49}$$

Comparing the left and right sides of this equation we can identify the four 4−vectors $W^{\mu 0}\dots W^{\mu 3}$, corresponding to the translation in α−direction. We combine them into a 4×4 tensor, $W^{\mu\alpha}(x) \equiv g^{\mu\alpha}\mathcal{L}(x)$. According to Eq. (1.41) this yields four Noether currents $j^{\mu 0}\dots j^{\mu 3}$ which we again combine into a tensor which is called *canonical energy-momentum tensor*

$$\boxed{T_c^{\mu\alpha}(x) = \pi^\mu(x)\partial^\alpha\Phi(x) - g^{\mu\alpha}\mathcal{L}(x)} \tag{1.50}$$

with the associated four conservation laws

$$\boxed{\partial_\mu T_c^{\mu\alpha}(x) = 0} \tag{1.51}$$

Separating the time and space components ($k = 1, 2, 3$) this system can be rewritten as

$$\frac{\partial}{\partial ct}T_c^{0\alpha}(x) + \frac{\partial}{\partial x^k}T_c^{k\alpha}(x) = 0, \tag{1.52}$$

or, splitting the α values into a time ($\alpha = 0$) and space ($\alpha = j = 1, 2, 3$) part,

$$\frac{\partial}{\partial ct}T_c^{00}(x) + \frac{\partial}{\partial x^k}T_c^{k0}(x) = 0, \tag{1.53}$$

$$\frac{\partial}{\partial ct}T_c^{0j}(x) + \frac{\partial}{\partial x^k}T_c^{kj}(x) = 0, \qquad j = 1, 2, 3, \tag{1.54}$$

where Eq. (1.54) is a system of three equations.

Equations (1.53) and (1.54) are four coupled local conservation laws connecting temporal changes of the tensor components T_c^{00} and T_c^{0j} with the divergence of spatial flux terms. As for any local conservation law we can find the associated global balance equation by integrating over space. Then we obtain

$$\frac{d}{dt}P^\alpha = 0, \quad \alpha = 0,1,2,3 \tag{1.55}$$

$$\text{with} \quad P^\alpha(t) = \int d^3r\, T_c^{0\alpha}(r,t), \tag{1.56}$$

where we transformed the flux term using the Gauss theorem

$$\int_V d^3r \frac{\partial}{\partial x^k} T_c^{k\alpha}(x) = \oint_{\partial V} dS_k T_c^{k\alpha}(x) = 0, \tag{1.57}$$

and assumed that the tensor components $T_c^{k\alpha}$ vanish at the system boundary ∂V. While equation (1.53) is the differential energy balance (local energy conservation law), equation (1.54) consitutes the local momentum conservation law for an arbitrary system described by the Lagrangian $\mathcal{L}$. This means we have obtained four scalar quantities P^α which are conserved.

What is remarkable about this result is its generality. We have not used any specific system property, we only used its Lagrangian given in terms of arbitrary fields Φ and considered infinitesimal space-time translations. It is obvious to guess that Equations (1.55) constitute the conservation laws of energy and momentum of the field Φ. We can readily verify this hypothesis by explicitly computing the quantities P^α. Inserting the result (1.50) into equation (1.55) we find

$$P^0 = \int d^3r \left\{\pi(x)\dot{\Phi}(x) - \mathcal{L}(x)\right\} \equiv \int d^3r\, \mathcal{H}(x) = H, \tag{1.58}$$

$$P^k = \int d^3r\, \pi(x)\partial^k\Phi(x), \tag{1.59}$$

where in the last line we took into account that $g^{0k} = 0$. The first line exactly coincides with our result (1.31) for the Hamilton density of the field Φ, i.e P^0 is nothing but the Hamilton function of the system and its conservation is the energy conservation law for the field Φ. Thus we can now establish the meaning of the individual components of the tensor $T_c^{k\alpha}$ and of the local conservation laws (1.53) and (1.54). The local energy balance (1.53) connects the time derivative of the Hamilton density T_c^{00} with the divergence of the energy current density – the vector with the components $(T_c^{10}, T_c^{20} T_c^{30})$. On the other hand, the momentum balance equation (1.54) relates the time change of the momentum density vector T_c^{0j} with the divergence of the stress tensor T_c^{kj}. In other words the $j-$column of T_c^{kj} is the momentum current density corresponding to the $j-$component of the momentum density T_c^{0j}.

We have considered only the simplest kind of symmetry transformation: space-time translations and established that homogeneity of space and time is related to conservation of total energy and momentum, respectively. Among other important symmetry transformation we mention rotations. One can demonstrate that isotropy of space-time is directly related to Lorentz invariance, see [WG93].

1.3 Field quantization

Let us now quantize the pair of canonically conjugate fields, $\Phi(\mathbf{r},t)$ and $\pi(\mathbf{r},t)$, just as it is done in quantum mechanics. To this end we replace the fields by operators

$$\Phi(\mathbf{r},t) \longrightarrow \hat{\Phi}(\mathbf{r},t), \tag{1.60}$$

$$\pi(\mathbf{r},t) \longrightarrow \hat{\pi}(\mathbf{r},t). \tag{1.61}$$

As in quantum mechanis, the two operators do not commute (Heisenberg uncertainty principle), and here we generalize the fundamental commutation relation, $[\hat{r}_i, \hat{p}_k] = i\hbar\delta_{i,k}$, to the case of functions of continuous arguments

$$\boxed{[\hat{\Phi}(\mathbf{r},t), \hat{\pi}(\mathbf{r}',t)]_{\mp} = i\hbar\,\delta(\mathbf{r}-\mathbf{r}')} \tag{1.62}$$

$$[\hat{\Phi}(\mathbf{r},t), \hat{\Phi}(\mathbf{r}',t)]_{\mp} = 0, \tag{1.63}$$

$$[\hat{\pi}(\mathbf{r},t), \hat{\pi}(\mathbf{r}',t)]_{\mp} = 0. \tag{1.64}$$

Thus identical fields always commute, while a field and its canonical momentum commute always, except for exactly the same space arguments in both operators. Also, the commutation relations hold only if both operators have the same time arguments. Finally, the subscript $\mp$ indicates an extension (beyond standard quantum mechanics) to bosonic (-) and fermionic (+) fields where we defined the commutator (anti-commutator) by

$$[\hat{A}, \hat{B}]_{\mp} \equiv \hat{A}\hat{B} \mp \hat{B}\hat{A}. \tag{1.65}$$

As in quantum mechanics we now apply the correspondence principle stating that functions of the canonical variables retain the same functional form. For example, the Hamilton density field now becomes an operator function of the same form as in classical field theory, and the same is true for the Lagrangian, the energy-momentum tensor and other functions.

$$\begin{aligned}
\mathcal{L}[\Phi,\dot{\Phi}] &\longrightarrow \hat{\mathcal{L}}[\hat{\Phi},\dot{\hat{\Phi}}], \\
\mathcal{H}[\Phi,\pi] &\longrightarrow \hat{\mathcal{H}}[\hat{\Phi},\hat{\pi}], \\
P^0 &\longrightarrow \hat{P}^0 = \int d^3r\, \hat{\mathcal{H}}(\mathbf{r}), \qquad (1.66) \\
P^k &\longrightarrow \hat{P}^k = \int d^3r\, \hat{\pi}(\mathbf{r})\partial^k\hat{\Phi}(\mathbf{r}). \qquad (1.67)
\end{aligned}$$

Thus the equations of motion following from the Euler-Lagrange equations will not change their form and be valid also for the operator fields. Similarly, the conservation laws of classical field theory, Eq. (1.51), remain formally valid for the operator of the energy-momentum tensor

$$\boxed{\partial_\mu \hat{T}_c^{\mu\alpha}(x) = 0} \qquad (1.68)$$

There is an additional property of the energy-momentum operator $\hat{P}^\alpha$ which we formulate as a theorem:

Theorem: *The operator $\hat{P}^k$ generates space translations in the direction r_k, with $k = 1, 2, 3$, and the operator $\hat{P}^0$ generates a time translation of the field $\hat{\Phi}$ of the following form*

$$[\hat{P}^k(t), \hat{\Phi}(\mathbf{r},t)]_- = \frac{\hbar}{i}\frac{\partial\hat{\Phi}(\mathbf{r},t)}{\partial r_k}, \qquad (1.69)$$

$$[\hat{P}^0(t), \hat{\Phi}(\mathbf{r},t)]_- = \frac{\hbar}{i}\frac{\partial\hat{\Phi}(\mathbf{r},t)}{\partial t}. \qquad (1.70)$$

Since $\hat{P}^0$ is nothing but the density of the Hamilton operator, equation (1.70) is a generalization of Heisenberg's equation for time-dependent operators to the case of field operators. Moreover, Eq. (1.69) further generalizes this equation to space derivatives.

Proof of relation (1.69):
The commutator (1.69) is straightforwardly transformed using the definition (1.67)

$$[\hat{P}^k(t), \hat{\Phi}(\mathbf{r},t)]_- = \int d^3r' \left[\hat{\pi}(\mathbf{r}',t)\frac{\partial\hat{\Phi}(\mathbf{r}',t)}{\partial r'_k}, \hat{\Phi}(\mathbf{r},t)\right]_- .$$

Transforming the commutator of a product according to $[AB, C] = A[B, C] + [A, C]B$ yields

$$\int d^3r' \left\{ \hat{\pi}(\mathbf{r}', t) \left[\frac{\partial \hat{\Phi}(\mathbf{r}', t)}{\partial r'_k}, \hat{\Phi}(\mathbf{r}, t) \right]_- + \left[\hat{\pi}(\mathbf{r}', t), \hat{\Phi}(\mathbf{r}, t) \right]_- \frac{\partial \hat{\Phi}(\mathbf{r}', t)}{\partial r'_k} \right\}.$$

The first term is zero due to the commutation of $\hat{\Phi}$ with itself, whereas the second term can be integrated using the commutation relation (1.62), and we immediately obtain the result (1.69). In a similar way, relation (1.70) is proven (see problem 2).

We may now combine the two relations (1.70) and (1.69) into a single four-vector relation

$$[\hat{P}^\alpha(t), \hat{\Phi}(x)]_- = \frac{\hbar}{i} \partial^\alpha \hat{\Phi}(x). \tag{1.71}$$

Furthermore, we may extend this property to an arbitrary function of the fields, $\hat{F}[\hat{\Phi}, \dot{\hat{\Phi}}, \hat{\Phi}']$, i.e.

$$[\hat{P}^\alpha(t), \hat{F}]_- = \frac{\hbar}{i} \partial^\alpha \hat{F}(x). \tag{1.72}$$

An interesting example is $\hat{F} \to \hat{P}^\mu$. Then Eq. (1.72) yields

$$[\hat{P}^\alpha(t), \hat{P}^\mu(t)]_- = \frac{\hbar}{i} \partial^\alpha \hat{P}^\mu(x) = 0. \tag{1.73}$$

This expression vanhishes because the energy-momentum $\hat{P}^\mu$, Eqs. (1.58), (1.59) is a conserved quantity and space-independent. Equation (1.73) shows that the energy-momentum $\hat{P}^\mu$ is part of the Poincare algebra, for more details see Ref. [WG93].

1.4 Phonons

As the first application of canonical field theory and field quantization we consider a continuous elastic medium which can perform oscillations around its equilibrium state. This generalizes our previous models, such as the one-dimensional chain or string, cf. Sections 2.4.1 and 2.5.1.

The state of the system is described by the displacement field $\mathbf{q}(\mathbf{r}, t)$ describing the local perturbation of the medium around the equilibrium position. The Lagrange density is the three-dimensional generalization of our previous $1d-$result (2.87),

$$\boxed{\mathcal{L}[\dot{\mathbf{q}}(\mathbf{r}, t), \mathbf{q}'(\mathbf{r}, t)] = \frac{\rho}{2} \sum_{1=1}^{3} \left\{ \left(\frac{\partial q_i(\mathbf{r}, t)}{\partial t} \right)^2 - c^2 \sum_{j=1}^{3} \left(\frac{\partial q_i(\mathbf{r}, t)}{\partial x_j} \right)^2 \right\} = \sum_{1=1}^{3} \mathcal{L}_i}$$

Here ρ is the mass density, $c = \sqrt{\frac{\sigma}{\rho}}$ is the propagation speed, and σ is the elastic tension. For the relation of these quantities to the discrete chain we refer to Section 2.5.1. Here our goal is primarily to demonstrate the application of the methods of field theory, so we will limit ourselves to the simplest example of elastic deformations: we will consider only longitudinal deformations and assume that σ is isotropic which leads to a direction-independent velocity c. Extensions to anisotropic systems and transverse excitations are straightforward.

1.4.1 Application of canonical field theory

Let us now apply our general field theory results to the present system.

1. We have to identify the general field variable Φ. Here the fields are replaced by a vector field $\mathbf{q}$ or three independent scalar fields q_1, q_2, q_3, i.e. $\Phi_l \longrightarrow q_i$, which we already indicated in the arguments of the Lagrange density (1.74).

2. Apply the general Euler-Lagrange equation (1.30) to the deformation field and evaluate the partial derivatives. For a fixed value $i = 1, 2, 3$ we have

$$\begin{aligned} 0 &= \frac{\delta \mathcal{L}_i}{\delta q_i} - \frac{d}{dt}\frac{\delta \mathcal{L}_i}{\delta \dot{q}_i} - \sum_{k=1}^{3} \frac{\partial}{\partial x^k} \frac{\delta \mathcal{L}_i}{\delta\left(\partial_{x_k} q_i\right)} \\ &= 0 - \rho \frac{d}{dt}\dot{q}_i + \rho c^2 \sum_{k,j=1}^{N} \frac{\partial}{\partial x^k}\frac{\partial q_i}{\partial x_j}\delta_{j,k} \\ &= \ddot{q}_i - c^2 \Delta q_i, \end{aligned} \qquad (1.74)$$

where, in the last line, we have canceled the common factor ρ. Thus we have obtained from the Euler-Lagrange equations a 3D wave equation, separately for each displacement component q_i.

3. We now calculate the canonical momentum, applying Eq. (1.28),

$$\pi_i = \frac{\delta \mathcal{L}_i}{\delta \dot{q}_i} = \rho \dot{q}_i, \qquad i = 1, 2, 3, \qquad (1.75)$$

where ϕ_i is the momentum density associated with the deformation q_i which generalizes the mechanical momentum p_i to continuous systems by the simple replacement $m \longrightarrow \rho$.

4. Next, we compute the Hamilton density, according to Eq. (1.32), where we have to eliminate $\dot{q}_i$ by the momentum, Eq. (1.75),

$$\begin{aligned}\mathcal{H} &= \sum_{i=1}^{3}\{\pi_i\dot{q}_i - \mathcal{L}_i\} \\ &= \frac{\rho}{2}\dot{\vec{q}}^{\,2} + \sum_{i,j=1}^{3}\frac{\rho c^2}{2}\left(\frac{\partial q_i}{\partial x_j}\right)^2 \\ &= \frac{\vec{\pi}^{\,2}}{2\rho} + \frac{\sigma}{2}\sum_{i,j=1}^{3}\left(\frac{\partial q_i}{\partial x_j}\right)^2 . \end{aligned} \tag{1.76}$$

Obviously, the first term is the kinetic energy density, whereas the sum contains all contributions to the potential energy arising from elastic tensions in the medium.

5. The energy-momentum tensor follows form the general definition (1.50), $\mu, \alpha = 0, 1, 2, 3$,

$$T_c^{\mu\alpha}(x) = \pi^{\mu}(x)\partial^{\alpha}\Phi(x) - g^{\mu\alpha}\mathcal{L}(x), \tag{1.77}$$

In particular, the total energy and momentum of the displacement field are obtained from Eqs. (1.58) and (1.59)

$$H(t) = \int d^3r \sum_{i=1}^{3}\{\pi_i(\mathbf{r},t)\dot{q}_i(\mathbf{r},t) - \mathcal{L}_i(\mathbf{r},t)\} = \int d^3r\, \mathcal{H}(\mathbf{r},t), \tag{1.78}$$

$$P^k(t) = \int d^3r \sum_{i=1}^{3}\pi_i(\mathbf{r},t)\partial^k q_i(\mathbf{r},t), \qquad k = 1,2,3 \tag{1.79}$$

These equations are the basis for the mechanics of elastic continuous media, including fluids and solids.

1.4.2 Expansion in terms of eigenfunctions

The solutions of the equation of motion (1.74) are oscillations or waves which depend on the initial and boundary conditions. Stationary solutions which solve Poisson's equations are standing waves with wave vector k. We may model an infinite system by considering a cube of side length L with volume $V = L^3$ and using periodic boundary conditions[5]. Then the solutions are given

[5]The limit of an infinite system, $L \to \infty$ can be performed at the end.

by

$$\mathbf{u}_\mathbf{k}(\mathbf{r}) = \mathbf{e}_\mathbf{k}\frac{e^{i\mathbf{kr}}}{L^{3/2}}, \qquad \mathbf{e}_\mathbf{k} = \frac{\mathbf{k}}{k}, \tag{1.80}$$

and, obviously, form a complete orthonormal system

$$\int_V d^3r\, \mathbf{u}_\mathbf{k}(\mathbf{r})\mathbf{u}^*_{\mathbf{k}'}(\mathbf{r}) = \delta_{\mathbf{k},\mathbf{k}'}. \tag{1.81}$$

Here we have chosen longitudinal polarization of the oscillations, i.e. $\mathbf{e}_\mathbf{k} \sim \mathbf{k}$. The system (1.80) forms a basis for arbitrary displacements,

$$\mathbf{q}(\mathbf{r},t) = \sum_\mathbf{k} B_\mathbf{k}\left\{b_\mathbf{k}(t)\mathbf{u}_\mathbf{k}(\mathbf{r}) + b^*_\mathbf{k}(t)\mathbf{u}^*_\mathbf{k}(\mathbf{r})\right\}, \tag{1.82}$$

where we included arbitrary time-dependent complex expansion coefficients and added the complex conjugate of the modes to assure that the displacement is real. The real coefficients $B_\mathbf{k}$ are introduced in order to adjust lateron the amplitude of the functions $b_\mathbf{k}(t)$ to one. Inserting the ansatz (1.82) into the wave equation (1.74)

$$0 = \sum_\mathbf{k} B_\mathbf{k}\left\{\left(\ddot{b}_\mathbf{k}(t) + c^2k^2b_\mathbf{k}(t)\right)\mathbf{u}_\mathbf{k}(\mathbf{r}) + \left(\ddot{b}^*_\mathbf{k}(t) + c^2k^2b^*_\mathbf{k}(t)\right)\mathbf{u}^*_\mathbf{k}(\mathbf{r})\right\},$$

yields a condition for the coefficients $b_\mathbf{k}(t)$. Since the functions $\mathbf{u}_\mathbf{k}$ form an orthonormal system, this equation can only be fulfilled if the terms in all parentheses vanish simultaneously, i.e. for all $\mathbf{k}$,

$$\ddot{b}_\mathbf{k}(t) + c^2k^2b_\mathbf{k}(t) = 0, \quad \text{with the solution} \quad b_\mathbf{k}(t) = b_{\mathbf{k}0}\, e^{-i\omega_k t}, \tag{1.83}$$

and the dispersion relation $\omega_k = c \cdot k$. By properly choosing the $B_\mathbf{k}$ we can always use $|b_{\mathbf{k}0}| = 1$ leaving open an arbitrary phase ϕ, i.e. $b_{\mathbf{k}0} = e^{i\phi_\mathbf{k}}$, which allows to fulfill the initial condition. Thus the final result for the displacement is

$$\mathbf{q}(\mathbf{r},t) = \frac{1}{L^{3/2}}\sum_\mathbf{k} \mathbf{e}_\mathbf{k}B_\mathbf{k}\left\{b_{\mathbf{k}0}e^{-i(\omega_k t - \mathbf{kr})} + b^*_{\mathbf{k}0}e^{i(\omega_k t - \mathbf{kr})}\right\}. \tag{1.84}$$

We can immediately write down the corresponding normal mode expansion of the canonically adjoint field, the momentum, Eq. (1.75), by differentiating the expansion (1.84) with respect to time,

$$\pi(\mathbf{r},t) = -i\frac{\rho}{L^{3/2}}\sum_\mathbf{k} \mathbf{e}_\mathbf{k}B_\mathbf{k}\omega_k\left\{b_{\mathbf{k}0}e^{-i(\omega_k t - \mathbf{kr})} - b^*_{\mathbf{k}0}e^{i(\omega_k t - \mathbf{kr})}\right\}. \tag{1.85}$$

Note the sign change in front of the adjoint contribution.

1.4.3 Quantization of the displacement field

Let us now quantize the canonical fields q_i and π_i with $i = x, y, z$ by introducing operators. We replace $q_i \longrightarrow \hat{q}_i$ and $\pi_i \longrightarrow \hat{\pi}_i$ and require that the standard canonical commutation relations are fulfilled,

$$[\hat{q}_i(\mathbf{r}, t), \hat{\pi}_l(\mathbf{r}', t)] = i\hbar\, \delta_{i,l}\, \delta(\mathbf{r} - \mathbf{r}'), \tag{1.86}$$

whereas the commutators of two identical fields are zero. Again, the two fields are taken at equal times. We now consider the normal mode representation (1.84) of the displacement field and the momentum, Eq. (1.85). Replacing here the fields by operators requires to introduce two independent operators (for each mode) also on the right hand sides. The only reasonable way of doing this is to replace the phase factors by operators, $b_{\mathbf{k}0} \longrightarrow \hat{b}_{\mathbf{k}}$ and $b^*_{\mathbf{k}0} \longrightarrow \hat{b}^\dagger_{\mathbf{k}}$. I.e. the normal mode expansion of the operators becomes

$$\begin{aligned} \hat{q}(\mathbf{r}, t) &= \frac{1}{L^{3/2}} \sum_{\mathbf{k}} \mathbf{e}_{\mathbf{k}} B_{\mathbf{k}} \left\{ \hat{b}_{\mathbf{k}} e^{-i(\omega_k t - \mathbf{k}\mathbf{r})} + \hat{b}^\dagger_{\mathbf{k}} e^{i(\omega_k t - \mathbf{k}\mathbf{r})} \right\}, &(1.87)\\ \hat{\pi}(\mathbf{r}, t) &= \frac{-i\rho}{L^{3/2}} \sum_{\mathbf{k}} \mathbf{e}_{\mathbf{k}} B_{\mathbf{k}} \omega_k \left\{ \hat{b}_{\mathbf{k}} e^{-i(\omega_k t - \mathbf{k}\mathbf{r})} - \hat{b}^\dagger_{\mathbf{k}} e^{i(\omega_k t - \mathbf{k}\mathbf{r})} \right\}. &(1.88) \end{aligned}$$

What remains is to establish the commutation relation between these operators. The criterion is, of course, that the original commutation relation (1.86) is satisfied. To this end we calculate the commutator of the i and l components of (1.87) and (1.88) and set it equal to the r.h.s. of (1.86),

$$\begin{aligned} i\hbar\, \delta_{i,l}\, \delta(\mathbf{r} - \mathbf{r}') &= \frac{-i\rho}{L^3} \sum_{\mathbf{k}\mathbf{k}'} \frac{k_i k'_l}{k k'} B_{\mathbf{k}} B_{\mathbf{k}'} \omega_{k'} \Big[\left\{ \hat{b}_{\mathbf{k}} e^{-i(\omega_k t - \mathbf{k}\mathbf{r})} + \hat{b}^\dagger_{\mathbf{k}} e^{i(\omega_k t - \mathbf{k}\mathbf{r})} \right\}, \\ &\quad \times \left\{ \hat{b}_{\mathbf{k}'} e^{-i(\omega_{k'} t - \mathbf{k}'\mathbf{r}')} - \hat{b}^\dagger_{\mathbf{k}'} e^{i(\omega_{k'} t - \mathbf{k}'\mathbf{r}')} \right\} \Big]. \end{aligned} \tag{1.89}$$

This equation can be satisfied by imposing, as an ansatz, the following bosonic commutation relations

$$[\hat{b}_{\mathbf{k}}, \hat{b}_{\mathbf{k}'}] = [\hat{b}^\dagger_{\mathbf{k}}, \hat{b}^\dagger_{\mathbf{k}'}] = 0\,, \tag{1.90}$$

$$[\hat{b}_{\mathbf{k}}, \hat{b}^\dagger_{\mathbf{k}'}] = \delta_{\mathbf{k},\mathbf{k}'}\,. \tag{1.91}$$

Then, from the four commutators in (1.89) only two involving $\hat{b}$ and $\hat{b}^\dagger$ remain (giving identical contributions, therefore, the factor 2) where $k' = k$, and we cancel the time-dependent exponents

$$\begin{aligned}\frac{\hbar}{\rho}\,\delta_{i,l}\,\delta(\mathbf{r}-\mathbf{r}') &= \frac{2}{V}\sum_{\mathbf{k}\mathbf{k}'}\frac{k_i k_l'}{kk'}B_{\mathbf{k}}B_{\mathbf{k}'}\omega_{k'}\left[\hat{b}_{\mathbf{k}},\hat{b}^\dagger_{\mathbf{k}}\right] \\ &\quad\times e^{-i(\omega_k-\omega_{k'})t}e^{i(\mathbf{kr}-\mathbf{k}'\mathbf{r}')} \qquad (1.92)\\ &= \frac{2}{V}\sum_{\mathbf{k}}\frac{k_i k_l}{k^2}B^2_{\mathbf{k}}\omega_k e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}')} \\ &\longrightarrow 2\int d^3k\,\frac{k_i k_l}{k^2}B^2_{\mathbf{k}}\omega_k e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}')}. \qquad (1.93)\end{aligned}$$

Since we consider a macroscopic system with $L \longrightarrow \infty$, the wave vector spectrum is quasi-continuous, and we could, in the last line, replace the sum $V^{-1}\sum_k$ by an integral over k_x, k_y, k_z, from minus to plus infinity. This integral vanishes if $i \neq l$, since the integrand of the k_i and k_l integrals is an odd function, which is in agreement with the Kronecker symbol on the l.h.s. On the other hand, in an isotropic medium, we expect that $B_{\mathbf{k}} = B_k$. Then, for $i = l$, the integral is independent of i, i.e. the integrals involving k_x^2, k_y^2 and k_z^2 are equal to each other and equal the integral containing $k^2/3$. Thus the k factors cancel and the $3d$ integral over $\mathbf{k}$ yields a delta function $\delta(\mathbf{r}-\mathbf{r}')$, again in agreement with the l.h.s. What is left to satisfy Eq. (1.93) is to properly choose the amplitudes $B_{\mathbf{k}}$, with the result

$$B_k = \left(\frac{\hbar}{2\times 3\rho\omega_k}\right)^{1/2} = \frac{L^{3/2}}{6^{1/2}}\,x_{0k}, \qquad (1.94)$$

where we have used $\rho = m/L^3$ and introduced the oscillator ground state wave function extension x_{0k} of mode k, $x_{0k} = (\hbar/m\omega)^{1/2}$. The final expression (1.94) shows that B_k has the dimension length to the power $5/2$ as it should be to guarantee the correct dimension (length) of the displacement q, cf. Eqs. (1.84) and (1.87).

With this we can write down the final expression for the operator of an arbitrary displacement in terms of the eigenfunctions of the elastic medium and their creation and annihilation operators,

$$\hat{q}(\mathbf{r},t) = \left(\frac{\hbar}{6\rho L^3}\right)^{1/2}\sum_{\mathbf{k}}\frac{\mathbf{e}_{\mathbf{k}}}{\omega_k^{1/2}}\left\{\hat{b}_{\mathbf{k}}e^{-i(\omega_k t-\mathbf{kr})}+\hat{b}^\dagger_{\mathbf{k}}e^{i(\omega_k t-\mathbf{kr})}\right\}. \qquad (1.95)$$

Finally we compute the Hamilton operator corresponding to the Hamilton function (1.76) and express it interms of the operators $\hat{b}_k$ and $\hat{b}_k^\dagger$. According to the correspondence principle, we rewrite Eq. (1.76) in terms of operators

$$\hat{H} = \int d^3r\left\{\frac{\hat{\vec{\pi}}^{\,2}}{2\rho}+\frac{\sigma}{2}\sum_{i,j=1}^{3}\left(\frac{\partial\hat{q}_i}{\partial x_j}\right)^2\right\}. \qquad (1.96)$$

Inserting now the normal mode expansions (1.87) and (1.88) we obtain (see Problem 3)

$$\hat{H} \;=\; \sum_{\mathbf{k}} \hbar\omega_k \left(\hat{n}_{\mathbf{k}} + \frac{1}{2}\right), \tag{1.97}$$

where $\hat{n}_{\mathbf{k}} = \hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}}$ is the number operator corresponding to mode $\mathbf{k}$. With expression (1.97) we have succeeded to diagonalize the hamiltonian of the elastic continuum and brought it to a very intuitive form: the hamiltonian is a superposition of independent quantized normal mode contributions, each having the form of a $1d$ linear harmonic oscillator with an occupation number operator and a zero point energy. This allows for a clear interpretation of the operators $\hat{b}^\dagger_{\mathbf{k}}$ and $\hat{b}_{\mathbf{k}}$: they create (annhilate) and elementary excitation characterized by the energy $\hbar\omega_k$ and momentum $\hbar\mathbf{k}$. These excitations are called *phonons.* The time and space dependence of the elastic deformation corresponding to the phonon mode $\mathbf{k}$ is given by a monochromatic plane wave, cf. Eq. (1.95).

In the $k-$sum we encounter the same problem as in Sec. 2.5.2: if the number of modes is infinite the sum diverges due to the infinite contribution of the ground state contribution. Thus, a cut-off is necessary (renormalization).

Time-dependent creation and annihilation operators. The operators $\hat{b}$ and $\hat{b}^\dagger$ which were introduced in Eqs. (1.87) and (1.88) were time-independent. We simply replaced the phase factors $b_{\mathbf{k}0}$ and $b^*_{\mathbf{k}0}$ by operators and left the time-dependence in the classical expressions (1.84) and (1.85) unchanged. Let us now discuss an alternative approach which leads to the same hamiltonian (1.97) and is frequently used. We then use new operators $\hat{b}(t)$ and $\hat{b}^\dagger(t)$ where the time-dependence is still open. Then the expansion of the displacement operator becomes

$$\hat{q}(\mathbf{r},t) \;=\; \left(\frac{\hbar}{2\cdot 3\rho L^3}\right)^{1/2} \sum_{\mathbf{k}} \frac{\mathbf{e}_{\mathbf{k}}}{\omega_k^{1/2}} \left\{\hat{b}_{\mathbf{k}}(t)\, e^{i\mathbf{k}\mathbf{r}} + \hat{b}^\dagger_{\mathbf{k}}(t)\, e^{-i\mathbf{k}\mathbf{r}}\right\}. \tag{1.98}$$

We can find the time-dependence, as in the classical case, by inserting (1.98) into the wave equation (1.74) which yields $\hat{b}_{\mathbf{k}}(t) = \hat{b}_{\mathbf{k}} e^{-i\omega_k t}$ and $\hat{b}^\dagger_{\mathbf{k}}(t) = \hat{b}^\dagger_{\mathbf{k}} e^{i\omega_k t}$. Alternatively, we can compute the time-derivative of the creation and annihilation operators by using the general Heisenberg equations of canonical field theory, Eq. (1.70) with the hamiltonian (1.97):

$$
\begin{aligned}
i\hbar\frac{d\hat{b}_{\mathbf{k}}}{dt} = \left[\hat{b}_{\mathbf{k}}, \hat{H}\right] &= \sum_{\mathbf{k}'} \hbar\omega_{k'} \left[\hat{b}_{\mathbf{k}}, \left(\hat{n}_{\mathbf{k}'} + \frac{1}{2}\right)\right] \\
&= \sum_{\mathbf{k}'} \hbar\omega_{k'} \left[\hat{b}_{\mathbf{k}}, \hat{b}^{\dagger}_{\mathbf{k}'}\hat{b}_{\mathbf{k}'}\right] \\
&= \sum_{\mathbf{k}'} \hbar\omega_{k'} \left\{\hat{b}^{\dagger}_{\mathbf{k}'} \left[\hat{b}_{\mathbf{k}}, \hat{b}_{\mathbf{k}'}\right] + \left[\hat{b}_{\mathbf{k}}, \hat{b}^{\dagger}_{\mathbf{k}'}\right] \hat{b}_{\mathbf{k}'}\right\} \\
&= \hbar\omega_k \hat{b}_{\mathbf{k}},
\end{aligned}
$$

where, in the last line, we used the commutation relations (1.90), (1.91). The solution of this equation yields the previous result $\hat{b}_{\mathbf{k}}(t) = \hat{b}_{\mathbf{k}} e^{-i\omega_k t}$.

1.5 Quantization of the free electromagnetic field. Photons

We now apply the formalism of canonical field theory to the electromagnetic field which is determined by Maxwell's equations. We will quantize the Maxwell field and will obtain the elementary excitations of the electromagnetic field – the photons.

1.5.1 Maxwell's equations. Electromagnetic potentials. Field tensor

Let us recall the basic quantities and equations of electrodynamics. The equations of motion of the electromagnetic field coupled to charges and currents are given by (2.10,2.11)

$$\begin{aligned} \mathrm{div}\mathbf{E}(\mathbf{r},t) &= 4\pi\rho(\mathbf{r},t), \quad \nabla\times\mathbf{E}(\mathbf{r},t) = -\frac{1}{c}\frac{\partial\mathbf{B}(\mathbf{r},t)}{\partial t}, &(1.99)\\ \mathrm{div}\mathbf{B}(\mathbf{r},t) &= 0, \qquad\qquad\quad \nabla\times\mathbf{B}(\mathbf{r},t) = \frac{4\pi}{c}\mathbf{j}(\mathbf{r},t) + \frac{1}{c}\frac{\partial\mathbf{E}(\mathbf{r},t)}{\partial t}. &(1.100)\end{aligned}$$

The charge and current densities ρ, j are not independent but are coupled via the charge conservation law (continuity equation)

$$\frac{\partial\rho(\mathbf{r},t)}{\partial t} + \mathrm{div}\,\mathbf{j}(\mathbf{r},t) = 0. \qquad (1.101)$$

All equations are local, i.e. all quantities appear with the same space and time arguments.

Two of Maxwell's equations (1.99,1.100) are satifsfied exactly by introducing the scalar and vector potentials, ϕ and $\mathbf{A}$, according to

$$\begin{aligned} \mathbf{B} &= \nabla\times\mathbf{A}, &(1.102)\\ \mathbf{E} &= -\nabla\phi - \frac{1}{c}\dot{\mathbf{A}}, &(1.103)\end{aligned}$$

where Eq. (1.102) solves the third of Maxwell's equations and Eq. (1.103) the second. In the remaining two equations the fields can be eliminated giving rise to

$$\begin{aligned} -\nabla^2\phi - \frac{1}{c}\nabla\dot{\mathbf{A}} &= 4\pi\rho, &(1.104)\\ \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right)\mathbf{A} &= \frac{4\pi}{c}\mathbf{j} - \frac{1}{c}\nabla\left(c\,\mathrm{div}\mathbf{A} + \dot{\phi}\right), &(1.105)\end{aligned}$$

where, in the second line, we used $\mathrm{rot}\,\mathrm{rot} = -\Delta + \mathrm{grad}\,\mathrm{div}$. The definition of the potentials is not unique. The equations for the measurable quantities – the electric and magnetic field strength E and B remain unchanged under any *gauge transformation* $\chi(\mathbf{r}, t)$ of the potentials

$$\mathbf{A} \longrightarrow \mathbf{A} + \nabla\chi, \tag{1.106}$$
$$\phi \longrightarrow \phi - \dot{\chi}. \tag{1.107}$$

Below we will use this gauge freedom to simplify the field equations. In particular, we will use the Coullomb gauge, $\nabla\mathbf{A} = 0$, and the Lorentz gauge. The idea of the latter is to eliminate the second term on the r.h.s. of Eq. (1.105), which is achieved by the following relation between A and ϕ

$$c\,\mathrm{div}\mathbf{A} + \dot{\phi} = 0. \tag{1.108}$$

This allows to transform the equations for the potentials into two decoupled wave equations,

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right)\phi = 4\pi\rho, \tag{1.109}$$
$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right)\mathbf{A} = \frac{4\pi}{c}\mathbf{j}. \tag{1.110}$$

This highly symmetric form is, of course, no conincidence but reflects relativistic covariance of the equations of the electromagnetic fields. This property can be made explicit by combining the scalar equation (1.109) and the vector equation (1.110) into a single equation by introducing the four-vector notation, as we did in section 1.1, but now for the electrical charge density and current density and for the potentials,

$$j^\mu = (c\rho, \mathbf{j}), \tag{1.111}$$
$$A^\mu = (\phi, \mathbf{A}). \tag{1.112}$$

Recalling the definition of the D'Alambert operator $\Box = \partial_\mu\partial^\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta$ we can rewrite the wave equations (1.109), (1.110)

$$\boxed{\Box A^\mu = \frac{4\pi}{c}j^\mu} \tag{1.113}$$

which is a covariant form of Maxwell's equations. Both sides are 4−vectors showing that this equation is Lorentz invariant. Analogously, we may introduce a 4−vector form for the continuity equation (1.101), the gauge transform

(1.106), (1.107) and the Lorentz gauge condition (1.108)

$$\begin{aligned}
\partial_\mu j^\mu &= 0, && (1.114)\\
A^\mu &\longrightarrow A^\mu - \partial^\mu \chi, \\
\partial_\mu A^\mu &= 0, && (1.115)
\end{aligned}$$

which are again Lorentz invariant.

While the Lorentz gauge is mathematically appealing, due to the symmetry between ϕ and $\mathbf{A}$, it partly masks the physical nature of the electromagnetic field suggesting the existence of four independent wave modes. However, this is not the case. As in the case of the displacement field, Sec. 1.4, there exist only three orthogonal wave excitations – one longitudinal (parallel to the wave vector $\mathbf{k}$) and two transverse ones (perpendicular to $\mathbf{k}$). This is also obvious from the Lorentz gauge (1.115) which couples the four components of A^μ, leaving only three of them independent. We will return to this question in Section 1.5.4 when we quantize the two transverse components of the electromagnetic field.

While the covariant formulation of the potentials is given by the 4−vector A^μ, the corresponding representation of the electric and magnetic field strengths is the *field tensor* (for more details see textbooks on electrodynamics, e.g. [LL62])

$$\boxed{F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix}} \qquad (1.116)$$

which is anti-symmetric and gauge-invariant and has the following properties $(i, j, k = 1, 2, 3)$

$$\begin{aligned}
F^{\mu\nu} &= -F^{\nu\mu}, && (1.117)\\
F^{k0} &= E_k, && (1.118)\\
F^{ij} &= -\epsilon^{ijk} B_k, && (1.119)\\
B_k &= -\frac{1}{2}\epsilon_{kij} F^{ij}, && (1.120)
\end{aligned}$$

where ϵ^{ijk} is the completely antisymmetric tensor [6].

[6] $\epsilon^{ijk} = 0$ unless i, j, k are different and $\epsilon^{ijk} = 1$ ($\epsilon^{ijk} = -1$) if $i = 1, j = 2, k = 3$ ($i = 1, j = 3, k = 2$) and analogously for cyclic permutations. The tensor ϵ_{ijk} has the same properties.

For completeness we give also the tensor with the lower indices which follows from the general tensor relations (1.6)

$$F_{\mu\nu} = g_{\mu\mu'} g_{\nu\nu'} F^{\mu'\nu'} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix}, \tag{1.121}$$

where $g_{\mu\mu'}$ is the metric tensor (1.4). $F_{\mu\nu}$ is also antisymmetric and differs from $F^{\mu\nu}$ only by the signs of the electric field components. The complete description of the electromagnetic field requires a second tensor, the so-called *dual tensor* $\tilde{F}^{\mu\nu}$ given by

$$\boxed{\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} = \begin{pmatrix} 0 & -B_1 & -B_2 & -B_3 \\ B_1 & 0 & E_3 & -E_2 \\ B_2 & -E_3 & 0 & E_1 \\ B_3 & E_2 & -E_1 & 0 \end{pmatrix}} \tag{1.122}$$

which follows from $F^{\mu\nu}$ by the so-called dual transform where the electric and magnetic fields are interchanged with the following sign change, $(\mathbf{E}, \mathbf{B}) \longrightarrow (\mathbf{B}, -\mathbf{E})$. The antisymmetric tensor $\epsilon^{\mu\nu\alpha\beta}$ is the generalization of $\epsilon^{\mu\nu\alpha}$. Verification of the components of the tensor $\tilde{F}^{\mu\nu}$ is subject of problem 4.

The two field tensors can be used to construct Lorentz invariants of the field which are of particular importance for the further analysis, in particular, for the construction of the Lagrange density and the action of the electromagnetic field in Sec. 1.5.2. In fact, one can show (see problems 5,6) that there exist two invariants,

$$\begin{aligned} F^{\mu\nu} F_{\mu\nu} &= 2\left(\mathbf{B}^2 - \mathbf{E}^2\right), && (1.123) \\ \tilde{F}^{\mu\nu} F_{\mu\nu} &= -4\mathbf{B} \cdot \mathbf{E}\,. && (1.124) \end{aligned}$$

The first is a Lorentz scalar and the second a pseudo-scalar [7]. Finally, we use the tensors to give another co-variant formulation of Maxwell's equations which follow from equation (1.113). In fact, differentiating the field tensor (1.116), we obtain

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= \partial_\mu \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) && (1.125) \\ &= \Box A^\nu - \partial^\nu \partial_\mu A^\mu \\ &= \Box A^\nu = \frac{4\pi}{c} j^\nu, \end{aligned}$$

[7] it involves a $3d$ scalar product rather than a scalar product of two $4-$vectors

where the second term in the second line vanishes due to the Lorentz gauge condition (1.115) and, in the last line, we have used Maxwell's equations (1.113). These are four equations since $\nu = 0, 1, 2, 3$. For $\nu = 0$, inserting the $0-$th column of the tensor, $F^{\mu 0}$, yields div$\mathbf{E} = 4\pi\rho$, whereas the equations for $\nu = 1, 2, 3$ yield $\nabla \times \mathbf{B} - \dot{\mathbf{E}}/c = 4\pi\mathbf{j}/c$, i.e. we recover two of Maxwell's equations.

Note that another differentiation of the l.h.s. of (1.125) yields zero, i.e. $\partial_\nu \partial_\mu F^{\mu\nu} = 0$ which is a consequence of definition (1.116) and of the Lorentz gauge. This requires that also the r.h.s. vanishes, i.e. $\partial_\nu j^\nu = 0$, showing that fulfilment of the continuity equation (1.114) by the charge and current density is a necessary condition for internal consistency of the electromagnetic field equations. The second pair of Maxwell's equations is given by the dual tensor:

$$\partial_\mu \tilde{F}^{\mu\nu} \;=\; 0. \tag{1.126}$$

When inserting the components of the dual tensor into this equations, we obtain, for $\nu = 0$, div$\mathbf{B} = 0$ and, for $\nu = 1, 2, 3$, $\nabla \times \mathbf{E} + \dot{\mathbf{B}}/c = 0$, i.e. we recover the remaining two equations. In contrast to the equations for $F^{\mu\nu}$ which couple to the electrical current density j^ν these equations have a zero r.h.s. This is a consequence of the incomplete symmetry between electric and magnetic field, arising from the nonexistence of magnetic monopoles ("magnetic charges").

1.5.2 Lagrange density of the free electromagnetic field

Let us now discuss how to construct the Lagrange density of the electromagnetic field. The Lagrange density $\mathcal{L}$ has the dimension of an energy density which, for the electromagnetic field, is well known and given by $(E^2 + B^2)/8\pi$. Furthermore, $\mathcal{L}$ has to be Lorentz invariant and gauge invariant. While gauge invariance would be satisfied if $\mathcal{L}$ depends only on E and B, Lorentz invariance is achieved if $\mathcal{L}$ is expressed by a combination of the two invariants (1.123) and (1.124). In fact, the first invariant (1.123) is sufficient and we rewrite it in terms of the field tensor

$$\mathcal{L} = \frac{1}{8\pi}\left(\mathbf{E}^2 - \mathbf{B}^2\right) = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu}. \tag{1.127}$$

The prefactor is arbitrary but, together with the factor 2 in Eq. (1.123), we may expect that $1/16\pi$ will yield the correct field energy. The minus sign is a guess which will lateron be confirmed by the correct sign of the field energy.

Our goal now is to express $\mathcal{L}$ in terms of the electromagnetic potentials A^μ, i.e. to find the functional form $\mathcal{L}[A^\mu, \partial_\mu A_\nu]$. Using the definition of the field tensor (1.116) we obtain from (1.127) and (1.116)

$$\begin{aligned}
-16\pi\mathcal{L} &= F^{\mu\nu}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) \\
&= F^{\mu\nu}\partial_\mu A_\nu - F^{\nu\mu}\partial_\mu A_\nu \\
&= 2F^{\mu\nu}\partial_\mu A_\nu,
\end{aligned}$$

where the second line was obtained by exchanging, in the second term, the indices ν and μ. [8] Thus, the final result is

$$\boxed{\mathcal{L} = -\frac{1}{8\pi}\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)\partial_\mu A_\nu}$$

showing that the Lagrange density does not depend on the potential, but only the derivative, $\mathcal{L} = \mathcal{L}[\partial_\mu A_\nu]$.

We now derive the equations of motion of the electromagnetic field from the general Euler-Lagrange equations (1.18). Substituting A_ν for the general field variable Φ_l, we obtain

$$0 = \frac{\delta\mathcal{L}}{\delta A_\nu} - \partial_\mu \frac{\delta\mathcal{L}}{\delta\partial_\mu A_\nu}, \qquad \nu = 0, \dots 3, \tag{1.128}$$

which is a system of four equations. The first term vanishes because $\mathcal{L}$ is independent of A_ν, whereas the second is most easily evaluated starting from expression (1.127)

$$-\frac{\delta\mathcal{L}}{\delta\partial_\mu A_\nu} = 2\,\frac{F^{\mu\nu}}{16\pi}\frac{\delta F_{\mu\nu}}{\delta\partial_\mu A_\nu} = \frac{F^{\mu\nu}}{4\pi}, \tag{1.129}$$

so the Euler-Lagrange equations become

$$\partial_\mu F^{\mu\nu} = 0, \tag{1.130}$$

i.e. we recover Maxwell's equations in the form (1.125). The zero of the r.h.s., i.e. missing current density, is due to the fact that we started from the Lagrange density of the free electromagnetic field. The generalization to nonzero charge density will be considered below in Sec. 1.6.

After having derived the field equations from the Lagrange density we can now use the Lagrange density to find the *canonical momentum* and the energy-momentum tensor of the electromagnetic field. Using the general definition (1.28) of the canonical momentum we can write, using the result (1.129),

$$\pi^{0\nu} = \frac{\delta\mathcal{L}}{\delta\partial_0 A_\nu} = -\frac{F^{0\nu}}{4\pi}, \qquad \nu = 0, 1, 2, 3. \tag{1.131}$$

[8] Recall that summation over the repeated indices μ and ν is implied, so interchanging the indices does not affect the result.

Inserting the 0−th row of the field tensor, we immediately obtain that $\pi^{00} = F^{00} = 0$, i.e. there exists no canonical momentum for the zero component of A^ν. In other words, $A^0 = \phi$ is not a dynamical variable of the electromagnetic field. The remaining three components of the momentum are given by $F^{0k},\ k = 1, 2, 3$, with the result that the momentum vector which is canonically adjoint to the vector potential $\mathbf{A}$ is given by the electric field:

$$\boxed{\vec{\pi}^0 = \frac{\mathbf{E}}{4\pi}} \tag{1.132}$$

The special role of the scalar potential is obvious from the fact that, by choosing a certain gauge, ϕ can be arbitrary.

Let us now derive the *energy-momentum tensor* of the electromagnetic field. We start from the general energy-momentum tensor of canonical field theory, Eq. (1.50),

$$T_c^{\mu\alpha}(x) = \pi^\mu(x)\partial^\alpha\Phi(x) - g^{\mu\alpha}\mathcal{L}(x).$$

Dropping the subscript "c", switching to the corresponding expression with the subscript α and replacing the scalar fields by four-vectors, $\Phi \to A_\nu$ and $\pi^\mu \to \pi^{\mu\nu} = -F^{\mu\nu}/4\pi$, generalizing the expression (1.131), we obtain [9]

$$T^\mu_\alpha = \pi^{\mu\nu}\partial_\alpha A_\nu - g^\mu_\alpha\mathcal{L} = -\frac{F^{\mu\nu}}{4\pi}\partial_\alpha A_\nu - g^\mu_\alpha\mathcal{L}.$$

In the first term on the right we now use the definition (1.116) of the field tensor,

$$\begin{aligned} F^{\mu\nu}\partial_\alpha A_\nu &= F^{\mu\nu}\left(F_{\alpha\nu} + \partial_\nu A_\alpha\right) \\ &= F^{\mu\nu}F_{\alpha\nu} + \partial_\nu\left(F^{\mu\nu}A_\alpha\right) - (-)[\partial_\nu F^{\nu\mu}]A_\alpha, \end{aligned} \tag{1.133}$$

where the last term vanishes due to Maxwell's equations. Also, the second term on the right, being a full 4-divergence, has no influence on the conservation laws and can be dropped. Then, using expression (1.127), we finally obtain

$$\begin{aligned} T^\mu_\alpha &= -\frac{F^{\mu\nu}}{4\pi}F_{\alpha\nu} + \frac{1}{16\pi}g^\mu_\alpha F^{\beta\nu}F_{\beta\nu} \\ &= -\frac{F^{\mu\nu}}{4\pi}F_{\alpha\nu} - \frac{1}{8\pi}g^\mu_\alpha\left(\mathbf{E}^2 - \mathbf{B}^2\right), \end{aligned} \tag{1.134}$$

where, in the last line, we used the field invariant (1.123). This tensor is symmetric, gauge invariant and Lorentz invariant, as it should be.

[9] Note that, when changing from superscript α to subscript α, the tensor elements transform according to Eqs. (1.8), i.e. $T^0_\alpha = T^{0\alpha}$ and $T^\beta_\alpha = -T^{\beta\alpha}$ for $\beta \neq 0$ and arbitrary α.

As it follows from canonical field theory, the energy-momentum tensor obeys the following four conservation laws

$$0 = \partial_\mu T^\mu_\alpha, \qquad \alpha = 0, 1, 2, 3, \tag{1.135}$$

and directly yields the energy density (Hamilton density)

$$\begin{aligned} \mathcal{H} = T^0_0 &= -\frac{F^{0\nu}}{4\pi} F_{0\nu} - \frac{1}{8\pi} g^0_0 \left(\mathbf{E}^2 - \mathbf{B}^2\right), \\ &= -\frac{1}{4\pi}\left(-E_1^2 - E_2^2 - E_3^2\right) - \frac{1}{8\pi}\left(\mathbf{E}^2 - \mathbf{B}^2\right) \\ &= \frac{1}{8\pi}\left(\mathbf{E}^2 + \mathbf{B}^2\right), \end{aligned} \tag{1.136}$$

and the momentum density of the field ($k-$component), see footnote 9,

$$S^k = -S_k = -T^0_k = \frac{F^{0\nu}}{4\pi} F_{k\nu},$$

where the second part of the tensor does not contribute. Using the properties of the tensor (1.118) and (1.119) we obtain

$$S^k = \frac{1}{4\pi}(-E_\nu)(-\epsilon_{k\nu j} B_j) = \frac{1}{4\pi}\left(\mathbf{E} \times \mathbf{B}\right)_k.$$

From this the total energy of the electromagnetic field (Hamilton function) and the total momentum (Poynting vector) are obtained by integration over the whole volume

$$H(t) = \frac{1}{8\pi}\int d^3r\, \mathcal{H}(\mathbf{r}, t) = \frac{1}{8\pi}\int d^3r \left(\mathbf{E}^2(\mathbf{r}, t) + \mathbf{B}^2(\mathbf{r}, t)\right), \tag{1.137}$$

$$\mathbf{P}(t) = \frac{1}{4\pi}\int d^3r\, c \cdot \mathbf{S}(\mathbf{r}, t) = \frac{c}{4\pi}\int d^3r\, \mathbf{E}(\mathbf{r}, t) \times \mathbf{B}(\mathbf{r}, t). \tag{1.138}$$

Using the results for the energy and momentum density, we can now explicitly write down the local conservation laws of energy and momentum (1.135), after multiplication with c

$$\frac{\partial}{\partial t}\mathcal{H} + \mathrm{div} c\mathbf{S} = 0 \tag{1.139}$$

$$\frac{\partial}{\partial t} S^k + \frac{\partial}{\partial x_j} c T^{jk} = 0, \qquad j, k = 1, 2, 3, \tag{1.140}$$

$$4\pi T^{jk} = -\left(E^j E^k + B^j B^k\right) + \frac{1}{2}\delta_{j,k}\left(\mathbf{E}^2 + \mathbf{B}^2\right), \tag{1.141}$$

where we identified the pressure tensor of the electromagnetic field T^{jk}.

Problem 7: Verify the expression (1.141) by direct evaluation of the tensor components (1.134).

1.5.3 Normal mode expansion of the electromagnetic field

We now wish to analyze the eigenmodes of the free electromagnetic field. These are the solutions of Maxwell's equations in vacuum with zero external charges and currents. The formal structure of the equations of motion depends on the chosen gauge: in the Lorentz gauge there exists a wave equation for the four-dimensional vector potential A^μ, cf. Eq. (1.113), however the electromagnetic field in vacuum is transverse, i.e. electric and magnetic field vectors oscillate perpendicular to the wave vector $\mathbf{k}$ (and orthogonal to each other). This means there exist only two possible polarizations of the electromagnetic field along the two axes in the plane perpendicular to $\mathbf{k}$. Furthermore, we have seen from the Lagrange formalism, cf. Sec. 1.5.2, that the scalar potential (the zero component of A^μ) is not a dynamical variable (there exists no canonical momentum associated with ϕ).

Thus, the Lorentz gauge masks the intrinsic properties of the field in vacuum and it, therefore, is advantageous to use the Coulomb gauge given by

$$\begin{aligned} \mathrm{div}\mathbf{A} &= 0, \qquad (1.142)\\ \phi &\equiv 0, \qquad (1.143) \end{aligned}$$

where the last line follows from the continuity equation (1.101), which yields $\dot{\phi} = 0$ and, in the absence of charges, we may set the potential equal to zero.

We now perform an expansion of the fields in terms of a complete set of eigenmodes. To this end we consider the case that the field is confined to a cube with side length L, i.e. the eigenmodes will be plane waves with two possible transverse polarizations, $\lambda = 1, 2$, which form a complete orthonormal set of functions

$$\begin{aligned} \mathbf{u}_{\lambda,\mathbf{k}}(\mathbf{r}) &= \frac{e^{i\mathbf{k}\mathbf{r}}}{L^{3/2}}\vec{e}_{\lambda,\mathbf{k}}, \qquad (1.144)\\ \sum_{\lambda=1,2} e^l_{\lambda,\mathbf{k}} e^j_{\lambda,\mathbf{k}} &= \delta_{l,j} - \frac{k_l k_j}{k^2}, \qquad (1.145)\\ \int d^3r\, \mathbf{u}^*_{\lambda,\mathbf{k}}(\mathbf{r}) \mathbf{u}_{\lambda',\mathbf{k}'}(\mathbf{r}) &= \delta_{\lambda,\lambda'}\delta_{\mathbf{k},\mathbf{k}'}. \end{aligned}$$

Eq. (1.145) is a compact mathematical form of the completeness and transversality condition which is easy to understand. Given the wave vector $\mathbf{k}$ we may decompose any vector $\mathbf{B}$ into a parallel and a perpendicular component by applying the parallel and transverse projection operator, respectively

$$\mathbf{B} = \mathbf{B}_{||} + \mathbf{B}_{\perp}, \tag{1.146}$$

$$\mathbf{B}_{||} = \hat{P}_{||}\mathbf{B} = \mathbf{k}\frac{\mathbf{k}\cdot\mathbf{B}}{k^2}, \tag{1.147}$$

$$\mathbf{B}_{\perp} = \hat{P}_{\perp}\mathbf{B} = (\hat{1} - \hat{P}_{||})\mathbf{B} = \mathbf{B} - \mathbf{k}\frac{\mathbf{k}\cdot\mathbf{B}}{k^2}, \tag{1.148}$$

which, in a Cartesian basis in wave number space[10], are represented by the tensors

$$P_{||}^{ij} = \frac{k_i k_j}{k^2}, \tag{1.149}$$

$$P_{\perp}^{ij} = \delta_{ij} - \frac{k_i k_j}{k^2}. \tag{1.150}$$

The system (1.144) forms a basis for arbitrary realizations of the fields A^μ and $\pi^\mu = -\dot{A}^\mu/(4\pi c)$. We introduce time-dependent expansion coefficients $b_{\lambda,\mathbf{k}}(t)$ and have to ensure that the fields are real functions:

$$\mathbf{A}(\mathbf{r},t) = \sum_{\mathbf{k}\,\lambda} B_{\lambda,\mathbf{k}} \left\{ b_{\lambda\mathbf{k}}(t)\mathbf{u}_{\lambda,\mathbf{k}}(\mathbf{r}) + b^*_{\lambda,\mathbf{k}}(t)\mathbf{u}^*_{\lambda,\mathbf{k}}(\mathbf{r}) \right\}, \tag{1.151}$$

The real coefficients $B_{\lambda,\mathbf{k}}$ are introduced in order to adjust lateron the amplitude of the functions $b_{\lambda,\mathbf{k}}(t)$ to one. Inserting the ansatz (1.151) into the wave equation (1.113) with $j^\mu = 0$

$$0 = \sum_{\lambda\,\mathbf{k}} B_{\lambda,\mathbf{k}} \left\{ \left(\ddot{b}_{\lambda,\mathbf{k}}(t) + c^2k^2 b_{\lambda,\mathbf{k}}(t) \right) \mathbf{u}_{\lambda,\mathbf{k}}(\mathbf{r}) + \left(\ddot{b}^*_{\lambda,\mathbf{k}}(t) + c^2k^2 b^*_{\lambda,\mathbf{k}}(t) \right) \mathbf{u}^*_{\lambda,\mathbf{k}}(\mathbf{r}) \right\},$$

yields a condition for the coefficients $b_{\lambda,\mathbf{k}}(t)$. Since the functions $\mathbf{u}_{\lambda,\mathbf{k}}$ form an orthonormal system, this equation can only be fulfilled if the terms in all parantheses vanish simultaneously, i.e. for all $\mathbf{k}$ and λ

$$\ddot{b}_{\lambda,\mathbf{k}}(t) + c^2k^2 b_{\lambda,\mathbf{k}}(t) = 0, \quad \text{with the solution} \quad b_{\lambda,\mathbf{k}}(t) = b_{\lambda,\mathbf{k}0}\, e^{-i\omega_k t}, \tag{1.152}$$

and the dispersion relation, $\omega_k = ck$, which is independent of the polarization λ. By properly choosing the $B_{\lambda,\mathbf{k}}$ we can always use $|b_{\lambda,\mathbf{k}0}| = 1$ leaving open an arbitrary phase ϕ, i.e. $b_{\lambda,\mathbf{k}0} = e^{i\phi_{\lambda,\mathbf{k}}}$, which allows to fulfill the initial condition. Thus the final result for the vector potential is

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{L^{3/2}} \sum_{\lambda\,\mathbf{k}} \mathbf{e}_{\lambda,\mathbf{k}} B_{\lambda,\mathbf{k}} \left\{ b_{\lambda,\mathbf{k}0} e^{-i(\omega_k t - \mathbf{kr})} + b^*_{\lambda,\mathbf{k}0} e^{i(\omega_k t - \mathbf{kr})} \right\}. \tag{1.153}$$

[10]For the representation in coordinate space, see Sec. 1.5.4

We can immediately write down the corresponding normal mode expansion of the canonically adjoint field, the momentum π, by differentiating the expansion (1.153) with respect to time,

$$\pi(\mathbf{r},t) = i\frac{1}{L^{3/2}}\sum_{\lambda\,\mathbf{k}} \mathbf{e}_{\lambda,\mathbf{k}}\frac{B_{\lambda,\mathbf{k}}\omega_k}{4\pi c}\left\{b_{\lambda,\mathbf{k}0}e^{-i(\omega_k t-\mathbf{kr})} - b^*_{\lambda,\mathbf{k}0}e^{i(\omega_k t-\mathbf{kr})}\right\}. \qquad (1.154)$$

Note the sign change in front of the adjoint contribution.

1.5.4 Quantization of the electromagnetic field

In Sec. 1.5.2 we have derived the canonical conjugate field variables for the free electromagnetic field – the 4-potential A^μ and the 4-momentum field. We now want to quantize these fields by replacing these quantities by operators and imposing the proper commutation relations. The quantization rules are

$$\begin{aligned}
A^\mu \quad &\longrightarrow \quad \hat{A}^\mu, & (1.155)\\
\pi^{0k} = \frac{1}{4\pi}E^k = -\frac{1}{4\pi c}\dot{A}^k \quad &\longrightarrow \quad \hat{\pi}^k = -\frac{1}{4\pi c}\frac{d}{dt}\hat{A}^k, \quad k=1,2,3, & (1.156)\\
\left[\hat{A}^k(\mathbf{r},t), \hat{\pi}^j(\mathbf{r}',t)\right] \quad &= \quad i\hbar\,\delta^T_{j,k}(\mathbf{r}-\mathbf{r}'), & (1.157)\\
\delta^T_{j,l}(\mathbf{r}) \quad &= \quad \int\frac{d^3k}{(2\pi)^3}\,P^{jl}_\perp(\mathbf{k})\,e^{i\mathbf{kr}}, & (1.158)
\end{aligned}$$

where the 0 component of π is zero. The commutation rules (1.157) are analogous to those of the displacement field, Sec. 1.3[11]. However, since there exist only two independent field components which are orthogonal to $\mathbf{k}$ we have replaced the three-dimensional delta function by the $2d$ transverse delta function which is the Fourier transform of the transverse projection tensor (1.150). By inserting the definition (1.150) into relation (1.158) we find an alternative representation,

$$\delta^T_{j,k}(\mathbf{r}) \quad = \quad \delta_{j,k}\delta(\mathbf{r}) + \frac{1}{4\pi}\partial_{r_j}\partial_{r_k}\frac{1}{r}. \qquad (1.159)$$

We now apply the normal mode expansions (1.153) and (1.154) to the quantized fields. Quantization in these expressions is performed by replacing

[11]We use bosonic commutation rules. This choice will be discussed at the end of this section

the expansion coefficients by operators, $b_{\lambda,\mathbf{k}0} \to \hat{b}_{\lambda,\mathbf{k}}$ and $b^*_{\lambda,\mathbf{k}0} \to \hat{b}^\dagger_{\lambda,\mathbf{k}}$, i.e.

$$\hat{\vec{A}}(\mathbf{r},t) = \frac{1}{L^{3/2}} \sum_{\lambda\,\mathbf{k}} \mathbf{e}_{\lambda,\mathbf{k}} B_{\lambda,\mathbf{k}} \left\{ \hat{b}_{\lambda,\mathbf{k}} e^{-i(\omega_k t - \mathbf{kr})} + \hat{b}^\dagger_{\lambda,\mathbf{k}} e^{i(\omega_k t - \mathbf{kr})} \right\}, \quad (1.160)$$

$$\hat{\vec{\pi}}(\mathbf{r},t) = \frac{i}{L^{3/2}} \sum_{\lambda\,\mathbf{k}} \mathbf{e}_{\lambda,\mathbf{k}} \frac{B_{\lambda,\mathbf{k}}\omega_k}{4\pi c} \left\{ \hat{b}_{\lambda,\mathbf{k}} e^{-i(\omega_k t - \mathbf{kr})} - \hat{b}^\dagger_{\lambda,\mathbf{k}} e^{i(\omega_k t - \mathbf{kr})} \right\}. \quad (1.161)$$

The expansion coefficient $\hat{b}^\dagger_{\lambda,\mathbf{k}}$ ($\hat{b}_{\lambda,\mathbf{k}}$) has the clear meaning of a creator (annihilator) of an elementary excitation of the electromagnetic field mode $(\lambda, \mathbf{k}, \omega_k)$, i.e. of a photon. What is left now is to obtain the commutation relations for these photon operators. To this end, we use the commutation relation (1.157) and insert the expansions (1.160) and (1.161) and divide by $i\hbar$

$$\sum_{\lambda\,\mathbf{k}\,\lambda'\,\mathbf{k}'} e^j_{\lambda,\mathbf{k}} e^l_{\lambda',\mathbf{k}'} \frac{B_{\lambda,\mathbf{k}} B_{\lambda',\mathbf{k}'}\omega_{k'}}{4\pi c\hbar L^3} \times \quad (1.162)$$

$$\left[\left\{ \hat{b}_{\lambda,\mathbf{k}} e^{-i(\omega_k t - \mathbf{kr})} + \hat{b}^\dagger_{\lambda,\mathbf{k}} e^{i(\omega_k t - \mathbf{kr})} \right\}, \left\{ \hat{b}_{\lambda',\mathbf{k}'} e^{-i(\omega'_k t - \mathbf{k}'\mathbf{r}')} - \hat{b}^\dagger_{\lambda',\mathbf{k}'} e^{i(\omega'_k t - \mathbf{k}'\mathbf{r}')} \right\} \right]$$

$$= \delta^T_{j,l}(\mathbf{r} - \mathbf{r}').$$

This equation is fulfilled by the following bosonic commutation rules

$$\left[\hat{b}_{\lambda,\mathbf{k}}, \hat{b}_{\lambda',\mathbf{k}'} \right] = \left[\hat{b}^\dagger_{\lambda,\mathbf{k}}, \hat{b}^\dagger_{\lambda',\mathbf{k}'} \right] = 0, \quad (1.163)$$

$$\left[\hat{b}_{\lambda,\mathbf{k}}, \hat{b}^\dagger_{\lambda',\mathbf{k}'} \right] = \delta_{\mathbf{k},\mathbf{k}'} \delta_{\lambda,\lambda'}. \quad (1.164)$$

Then Eq. (1.162) reduces to

$$\delta^T_{j,l}(\mathbf{r} - \mathbf{r}') = \quad (1.165)$$

$$-2 \sum_{\lambda\,\mathbf{k}\,\lambda'\,\mathbf{k}'} e^j_{\lambda,\mathbf{k}} e^l_{\lambda',\mathbf{k}'} \frac{B_{\lambda,\mathbf{k}} B_{\lambda',\mathbf{k}'}\omega_{k'}}{4\pi c\hbar L^3} \delta_{\mathbf{k},\mathbf{k}'} \delta_{\lambda,\lambda'} \times \quad (1.166)$$

$$e^{i(\omega'_k - \omega_k)t} e^{-i(\mathbf{k}'\mathbf{r}' - \mathbf{kr})} =$$

$$\sum_{\mathbf{k}} e^{-i\mathbf{k}(\mathbf{r}' - \mathbf{r})} \sum_{\lambda} e^j_{\lambda,\mathbf{k}} e^l_{\lambda,\mathbf{k}} \frac{-B^2_{\lambda,\mathbf{k}}\omega_k}{2\pi c\hbar L^3}. \quad (1.167)$$

According to the completeness relation (1.145), the sum over λ yields the transverse projector $P^{jl}_\perp$, provided there is no $\lambda-$dependent coefficient. Finally, the k-sum is just the Fourier transform of $P^{jl}_\perp$ which is nothing but the transverse delta function, cf. Eq. (1.158), if there is no $k-$dependent coefficient. The requirements on the coefficients are readily fulfilled by choosing $B_k = (-2\pi\hbar c L^3/\omega_k)^{1/2}$.

Thus we have succeeded in finding the normal mode representation of the field operators of the electromagnetic field which obey bosonic commutation rules. The final result for the field operators which follows from inserting the coefficients into (1.160,1.161) is given by

$$\hat{\vec{A}}(\mathbf{r},t) = \sqrt{2\pi\hbar c}\sum_{\lambda\,\mathbf{k}}\frac{\mathbf{e}_{\lambda,\mathbf{k}}}{\omega_k^{1/2}}\left\{\hat{b}_{\lambda,\mathbf{k}}e^{-i(\omega_k t-\mathbf{kr})}+\hat{b}^\dagger_{\lambda,\mathbf{k}}e^{i(\omega_k t-\mathbf{kr})}\right\}, \tag{1.168}$$

$$\hat{\vec{\pi}}(\mathbf{r},t) = i\sqrt{\frac{\hbar}{8\pi c}}\sum_{\lambda\,\mathbf{k}}\mathbf{e}_{\lambda,\mathbf{k}}\omega_k^{1/2}\left\{\hat{b}_{\lambda,\mathbf{k}}e^{-i(\omega_k t-\mathbf{kr})}-\hat{b}^\dagger_{\lambda,\mathbf{k}}e^{i(\omega_k t-\mathbf{kr})}\right\}. \tag{1.169}$$

Finally, using the normal mode (photon) representation of the field operators we can transform all second quantization operators to the photon picture. In particular, we obtain for the hamiltonian (1.137)

$$\begin{aligned}\hat{H}(t) &= \frac{1}{8\pi}\int d^3r\left(\hat{\mathbf{E}}^2(\mathbf{r},t)+\hat{\mathbf{B}}^2(\mathbf{r},t)\right)\\ &= \int d^3r\left\{2\pi\hat{\vec{\pi}}^2(\mathbf{r},t)+\frac{1}{8\pi}\left(\vec{\nabla}\times\hat{\vec{A}}\right)^2(\mathbf{r},t)\right\}\\ &= \sum_{\lambda\,\mathbf{k}}\hbar\omega_k\left(\hat{b}^\dagger_{\lambda,\mathbf{k}}\hat{b}_{\lambda,\mathbf{k}}+\frac{1}{2}\right).\end{aligned} \tag{1.170}$$

Thus, the hamiltonian of the free electromagnetic field is a superposition of linear harmonic oscillator hamiltonians corresponding to the quantized normal modes of the field. For details of the derivation of the final result, see problem 8.

Let us briefly discuss our result.

i) We have quantized the free electromagnetic field which is transverse with two independent components (polarizations).

ii) Type and number of independent field modes (normal modes) is the same as in the classical description. In an infinite systems, the normal modes are monochromatic plane waves, in other geometries the modes follow from the solution of the wave equation with the proper boundary conditions.

iii) Other examples would be a small radiation source which emittes in all directions. In this case the normal modes would be spherical waves. Furthermore, in a finite system, the normal modes depend on the properties of the boundaries such as the reflectivity of the walls or losses due to dissipation. Special resonators are used to select a certain number of modes, thus reducing the $k-$sum (wave guides, lasers etc.).

iv) The field consists of an integer number of elementary quanta (photons) with an elementary energy $\hbar\omega_k$ and a momentum $\hbar\mathbf{k}$. Experiments confirm that each mode can be multiply excited, i.e. contain a large number $N_{\mathbf{k},\lambda}$ of modes. This would not be possible if photons would obey Fermi statistics and, hence, the Pauli principle. Thus the choice of bosonic commutation rules (1.157) is justified.

v) In second quantization the electromagnetic field is described by the field operators $\hat{A}$ and $\hat{\pi}$ which are randomly fluctuating quantities. This transforms into randomly fluctuating mode amplitudes $\hat{b}_{\mathbf{k},\lambda}$, $\hat{b}^\dagger_{\mathbf{k},\lambda}$ or their combinations such as the number operator $\hat{n}_{\mathbf{k},\lambda} = \hat{b}^\dagger_{\mathbf{k},\lambda}\hat{b}_{\mathbf{k},\lambda}$. Measurements will yield the eigenvalues, i.e. the number of photons of a given mode, $n_{\mathbf{k},\lambda} = 0, 1, 2, \dots$

vi) Other measurable quantities are expectation values of the operators $\hat{A}$ and $\hat{\pi}$ or of operator products (correlation functions). To compute the expectation values requires knowledge of the statistical properties of the field. This can, for example, be a quantum mechanical state (pure state) which is being prepared in an experiment (such as in quantum optics)

vii) Alternatively, the field can be in a mixed state such as in thermodynamic equilibrium. Then the theory requires averaging of operator products with a thermodynamic density operator which leads to a statistical occupation of the photon modes with the mean occupations given by a Bose distribution.

viii) The most general situation corresponds to a nonequilibrium state of the quantized field which can be caused by an external perturbation. Then the mean occupation of each mode may change in time, $n_{\mathbf{k},\lambda} = n_{\mathbf{k},\lambda}(t)$, and the question arises how to theoretically describe this evolution. The most general theory which solves this problem is the theory nonequilibrium Green functions (here, the photon Green function), which considers expectation values of operator products of the type $\langle \hat{b}_{\mathbf{k},\lambda}(t)\hat{b}^\dagger_{\mathbf{k}',\lambda'}(t')\rangle$, see e.g. Ref. [Bon98].

Finally, we note that, with the second quantization of the electromagnetic field and the introduction of photons, we have found a consistent concept for the description of the field – in a sense, this is the modern picture of the particle-field duality, originally introduced by Max Planck the founder of quantum theory. Indeed, this theory contains wave properties – here in terms of the normal modes - and particles. Only the latter are not to be understood as particles in a mechanical sense but as elementary excitations of the field modes – the photons – which carry an energy quantum $\hbar\omega_k$ and a momentum quantum $\hbar\mathbf{k}$[12].

1.6 Quantization of the electromagnetic field coupled to charged particles

So far we have considered the free electromagnetic field and its quantization. Let us now generalize these results to the case that the field is coupled to charged particles. We will generalize the Lagrange formalism to the full field-matter interaction and find the corresponding Euler-Lagrange equations. The first question we have to solve is how to incroporate the coupling between particles and fields into the minimal action principle. As with the choice of the Lagrange function, cf. Sec. 1.1 there is, in principle, large freedom in constructing the action for particles and fields. We will again follow the rule to prefer the simplest choice possible, see also the discussion in Ref. [LL62].

The total action of the coupled particle-field system can always be written as the sum of three terms

$$\boxed{S = S_F + S_M + S_{MF}} \tag{1.171}$$

where we denote the field (matter) term which contains only field (matter) variables by "F" ("M") and the coupling term by "MF". The first term is the action of the free electromagnetic field which was studied in Sec. 1.5.2. We are now going to analyze the second term.

1.6.1 Lagrangian of a classical relativistic particle

The total action of a system of N charged particles is additive $S_M = \sum_{i=1}^{N} S_{Mi}$. This is, because the Coulomb interaction of the particles is mediated by the electromagnetic field, and this effect will be included in the action S_{MF}.

Thus, let us start by consdiering a single classical free non-relativistic particle "i" with mass m_i and recall its action. Since the only energy contribution is due to kinetic energy T, the Lagrange function coincides with T, cf. Sec. 1.1.1, $L_i^0 = m_i v^2/2$. Obviously, this expression is not Lorentz invariant. To generalize

[12] Interestingly, Planck himself maintained serious doubts about the consistency of his own theory [?]. Considering, as an example, a spherical wave with an intensity decreasing as r^{-2} where r is the distance from the source he could not imagine how to deal with the situation when the intensity falls below that corresponding to a single photon. This is naturally solved in the standard probabilistic interpretation of quantum mechanics.

L_i^0 to a relativistic particle we need to start from Lorentz scalars. The simplest mechanical Lorentz scalar is the space time interval, the square of which is $(ds_i)^2 = c^2(dt)^2 - (d\mathbf{r}_i)^2$, cf. Sec. 1.1. Then, the action for a macroscopic physical process between points "a" and "b" involving particle "i" is

$$\begin{aligned} S_{Mi} &= \alpha_i \int_a^b ds = \alpha_i c \int_{t_a}^{t_b} \sqrt{1 - \frac{\mathbf{v}_i^2}{c^2}} dt \\ &\equiv \int_{t_a}^{t_b} dt\, L_{Mi}, \end{aligned} \tag{1.172}$$

where we used $\mathbf{v}_i = d\mathbf{r}_i/dt$, and the constant α_i is yet to be determined. In the last line we used the general definition of the Lagrange function which yields, for a relativistic point particle, $L_{Mi} = \alpha_i c\sqrt{1 - \frac{\mathbf{v}_i^2}{c^2}}$. The constant is readily determined by requiring that the known non-relativistic limit is recovered, i.e.

$$L_i^0(\mathbf{v}_i) = \frac{m_i}{2}\mathbf{v}_i^2 = \lim_{v_i/c\to 0} L_{Mi}(\mathbf{v}_i) = \alpha_i c\left(1 - \frac{1}{2}\frac{\mathbf{v}_i^2}{c^2}\right).$$

Comparison of the left and right sides and dropping the velocity independent first term yields the constant to be $\alpha_i = -m_i c$ with the result for the Lagrange function of a relativistic classical point particle

$$\boxed{L_{Mi}(\mathbf{v}_i) = -m_i c^2 \sqrt{1 - \frac{\mathbf{v}_i^2}{c^2}}} \tag{1.173}$$

Applying the standard formulas of the Lagrange formalism, from this we immediately recover the momentum

$$\mathbf{p}_i = \frac{\partial L_{Mi}}{\partial \mathbf{v}_i} = \frac{m_i \mathbf{v}_i}{\sqrt{1 - \frac{\mathbf{v}_i^2}{c^2}}}, \tag{1.174}$$

and the total (kinetic) energy

$$E_i = \mathbf{p}_i \mathbf{v}_i - L_{Mi} = \frac{m_i c^2}{\sqrt{1 - \frac{\mathbf{v}_i^2}{c^2}}}. \tag{1.175}$$

Now, the hamiltonian follows from E_i, as usual, by eliminating the velocity. Squaring (1.175), we obtain

$$\begin{aligned} \frac{E_i^2}{c^2} &= \frac{m_i^2 c^2}{1 - \frac{v_i^2}{c^2}} = \frac{m_i^2 \mathbf{v}_i^2}{1 - \frac{\mathbf{v}_i^2}{c^2}} + \frac{m_i^2 (c^2 - \mathbf{v}_i^2)}{1 - \frac{\mathbf{v}_i^2}{c^2}} \\ &= \mathbf{p}_i^2 + m_i c^2, \end{aligned}$$

which yields the familiar relativistic energy-momentum dispersion

$$\boxed{H_{Mi}(\mathbf{p}_i) = \sqrt{c^2\mathbf{p}_i^2 + m_i^2 c^4}} \tag{1.176}$$

Finally, the Lagrange formalism immediately yields the (trivial) equation of motion (the Euler-Lagrange equation) of a free particle:

$$0 = -\frac{\partial L_{Mi}}{\partial \mathbf{q}_i} + \frac{d}{dt}\frac{\partial L_{Mi}}{\partial \mathbf{v}_i} = \frac{d}{dt}\mathbf{p}_i. \tag{1.177}$$

To shorten the notation, below we will denote the relativistic square root by $\gamma_i = (1 - \mathbf{v}_i^2/c^2)^{-1/2}$.

1.6.2 Relativistic particle coupled to the electromagnetic field

Let us now analyze how the equation of motion of the particle changes if it interacts with an electromagnetic field. We then have to re-derive the Euler-Lagrange equation starting from the total action $S = S[\mathbf{q}_i, \mathbf{v}_i; A^\mu]$, Eq. (1.171), and performing the variation with respect to the particle variables. Due to the linear relation between action and Lagrange function, the latter will also consist of three parts from which the field part is, by definition, independent of the particle coordinates. Therefore, the equation of motion of particle "i" follows from the Lagrange function $L_{Mi} + L_{Mi,F}$ generalizing Eq. (1.177)

$$\begin{aligned} 0 &= -\frac{\partial L_{Mi} + L_{Mi,F}}{\partial \mathbf{q}_i} + \frac{d}{dt}\frac{\partial L_{Mi} + L_{Mi,F}}{\partial \mathbf{v}_i} \\ &= -\frac{\partial L_{Mi,F}}{\partial \mathbf{q}_i} + \frac{d}{dt}\mathbf{p}_i + \frac{d}{dt}\frac{\partial L_{Mi,F}}{\partial \mathbf{v}_i} \end{aligned} \tag{1.178}$$

Inserting the result for the relativistic momentum, Eq. (1.174), we obtain the relativistic generalization of Newton's equation

$$\boxed{\frac{d}{dt}\frac{m_i\mathbf{v}_i}{\sqrt{1-\frac{\mathbf{v}_i^2}{c^2}}} = \frac{\partial L_{Mi,F}}{\partial \mathbf{q}_i} - \frac{d}{dt}\frac{\partial L_{Mi,F}}{\partial \mathbf{v}_i} = \mathbf{F}_{Li}} \tag{1.179}$$

where the r.h.s. is the relativistic Lorentz force which is determined by the field-matter interaction. While the explicit form of the interaction Lagrangian $L_{Mi,F}$ is still open it is clear that, in the equation of motion (1.179) for our classical point particle, the r.h.s. has to be taken (after differentiation) at the current phase space point $\{\mathbf{q}_i(t), \mathbf{v}_i(t)\}$ of the particle.

1.6.3 Lagrangian of charged particles in an EM field

Let us now consider how to describe, in the Lagrange formalism, the interaction between a charged particle with the electromagnetic field. Again, we choose the simplest possible form. The key quantities describing the field matter interaction are the particle charge (and eventuall the charge current density) whereas the field is represented by the potential A^μ. The simplest way to couple both is to include both factors in first order. To do this in a Lorentz invariant way we need a scalar product of two 4−vectors which we form from A^μ and x^μ. The proper choice of the action is (the pre-factor is chosen to correctly reproduce the interaction energy)

$$\begin{aligned} S_{Mi,F} &= -\int_a^b \frac{e_i}{c} A_\mu dx^\mu = -\frac{e_i}{c}\int_a^b \{c\phi dt - \mathbf{A} d\mathbf{r}\} \\ &= -\frac{e_i}{c}\int_{t_a}^{t_b} \{c\phi - \mathbf{A}\mathbf{v}_i\}\, dt, \end{aligned} \tag{1.180}$$

which yields the following result for the field-matter Lagrangian

$$\boxed{L_{Mi,F}[\mathbf{v}_i(t); A^\mu(\mathbf{q}_i,t)] = \frac{e_i}{c}\mathbf{A}\mathbf{v}_i - e_i\phi} \tag{1.181}$$

where the potentials have to be taken at the current position of the particle.

With this result we can derive the canonical momentum and the Hamilton function of the particle in the electromagnetic field replacing, in our previous expressions for the free particle, $L_{Mi} \to L_{Mi} + L_{Mi,F} \equiv L_i$. The momentum is then

$$\mathbf{P}_i = \frac{\partial L_i}{\partial \mathbf{v}_i} = \mathbf{p}_i + \frac{e_i}{c}\mathbf{A}(\mathbf{q}_i) = m_i\mathbf{v}_i\gamma_i + \frac{e_i}{c}\mathbf{A}(\mathbf{q}_i), \tag{1.182}$$

where $\mathbf{p}_i$ is given by Eq. (1.174). Analogously, we obtain the energy

$$E_i = \mathbf{p}_i\mathbf{v}_i - L_i = m_i c^2\gamma_i + e_i\phi(\mathbf{q}_i). \tag{1.183}$$

To obtain the Hamilton function we again eliminate the velocity, as was done with Eq. (1.175),

$$\frac{(E_i - e_i\phi)^2}{c^2} = \mathbf{p}_i^2 + m_i c^2.$$

The hamiltonian of a relativistic particle should be written as a function of the canonical momentum which now is given by Eq. (1.182). Thus solving for E_i and expressing $\mathbf{p}_i$ by $\mathbf{P}_i$ we obtain

$$\boxed{H_i(\mathbf{q}_i,\mathbf{P}_i) = \sqrt{c^2\left(\mathbf{P}_i - \frac{e_i}{c}\mathbf{A}(\mathbf{q}_i)\right)^2 + m_i^2c^4} + e_i\phi(\mathbf{q}_i)} \tag{1.184}$$

For completeness, we consider the non-relativistic limit of the above results. Then $L_{Mi} \to L_i^0$ but $L_{Mi,F}$ remains unchanged, i.e. $L_i \to \frac{m_i}{2}\mathbf{v}_i^2 + \frac{e_i}{c}\mathbf{A}\mathbf{v}_i - e_i\phi$. Consequently, $\mathbf{P}_i = \frac{\partial L_i}{\partial \mathbf{v}i} = m_i\mathbf{v}_i - \frac{e_i}{c}\mathbf{A}$, and the non-relativistic hamiltonian reduces to

$$H_i(\mathbf{q}_i, \mathbf{P}_i) = \frac{1}{2m_i}\left(\mathbf{P}_i - \frac{e_i}{c}\mathbf{A}(\mathbf{q}_i)\right)^2 + e_i\phi(\mathbf{q}_i).$$

With the result for the field-matter interaction Lagrangian we can now evaluate the Lorentz force (1.179) on a relativistic particle. The result is the same as in the non-relativistic case, familiar from electrodynamics,

$$\mathbf{F}_{Li}(t) = e_i\mathbf{E}(\mathbf{q}_i, t) + \frac{e_i}{c}\mathbf{v}_i \times \mathbf{B}(\mathbf{q}_i, t) \tag{1.185}$$

Proof: To evaluate the expression

$$\mathbf{F}_{Li} = \frac{\partial L_{Mi,F}}{\partial \mathbf{q}_i} - \frac{d}{dt}\frac{\partial L_{Mi,F}}{\partial \mathbf{v}_i}, \tag{1.186}$$

we begin with transforming the second term taking into account that, for a continuous vector field $\mathbf{A}(\mathbf{r}, t)$, the total time derivative is $\frac{d}{dt}\mathbf{A} = \{\frac{\partial}{\partial t} + (\mathbf{v}\vec{\nabla})\}\mathbf{A}$. Recalling further the relation between electric field and vector potential, $\mathbf{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}$, we can transform

$$\begin{aligned} -\frac{d}{dt}\frac{\partial L_{Mi,F}}{\partial \mathbf{v}_i} &= -\frac{e_i}{c}\frac{d}{dt}\mathbf{A} = -\frac{e_i}{c}\left\{\frac{\partial}{\partial t} + (\mathbf{v}_i\vec{\nabla})\right\}\mathbf{A} \\ &= e_i(\mathbf{E} + \vec{\nabla}\phi) - \frac{e_i}{c}(\mathbf{v}_i\vec{\nabla})\mathbf{A}. \end{aligned} \tag{1.187}$$

For the transformation of the first term in (1.186) we recall the vector identities for an arbitrary pair of vectors

$$\begin{aligned} \vec{\nabla}(\mathbf{C}\cdot\mathbf{D}) &= (\mathbf{C}\cdot\vec{\nabla})\mathbf{D} + (\mathbf{D}\cdot\vec{\nabla})\mathbf{C} + \\ &+ \mathbf{D}\times(\vec{\nabla}\times\mathbf{C}) + \mathbf{C}\times(\vec{\nabla}\times\mathbf{D}), \end{aligned}$$

which allows to transform

$$\frac{\partial}{\partial \mathbf{q}_i}\left[\mathbf{A}(\mathbf{q}_i)\mathbf{v}_i\right] = \left(\mathbf{v}_i\cdot\frac{\partial}{\partial \mathbf{q}_i}\right)\mathbf{A}(\mathbf{q}_i) + \mathbf{v}_i\times\left(\frac{\partial}{\partial \mathbf{q}_i}\times\mathbf{A}(\mathbf{q}_i)\right),$$

where we took into account that $\mathbf{v}_i$ does not depend on $\mathbf{q}_i$. Now we can transform

$$\begin{aligned} \frac{\partial L_{Mi,F}}{\partial \mathbf{q}_i} &= \frac{e_i}{c}\frac{\partial}{\partial \mathbf{q}_i}\left[\mathbf{A}(\mathbf{q}_i)\mathbf{v}_i\right] - e_i\frac{\partial}{\partial \mathbf{q}_i}\phi(\mathbf{q}_i) \qquad (1.188) \\ &= \frac{e_i}{c}\left(\mathbf{v}_i\cdot\frac{\partial}{\partial \mathbf{q}_i}\right)\mathbf{A}(\mathbf{q}_i) + \frac{e_i}{c}\mathbf{v}_i\times\left(\frac{\partial}{\partial \mathbf{q}_i}\times\mathbf{A}(\mathbf{q}_i)\right) - e_i\frac{\partial}{\partial \mathbf{q}_i}\phi(\mathbf{q}_i). \end{aligned}$$

Now we can add up the two terms (1.187) and (1.188) and obtain, after cancellations,

$$-\frac{d}{dt}\frac{\partial L_{Mi,F}}{\partial \mathbf{v}_i} + \frac{\partial L_{Mi,F}}{\partial \mathbf{q}_i} = e_i \mathbf{E}(\mathbf{q}_i) + \frac{e_i}{c}\mathbf{v}_i \times \left(\frac{\partial}{\partial \mathbf{q}_i} \times \mathbf{A}(\mathbf{q}_i)\right), \qquad (1.189)$$

which is just the Lorentz force (1.186) acting on point particle "i".

With this the treatment of the particle dynamics is completed. Using the Lagrange formalism with a Lagrangian for a relativistic point particle and an additional field-matter interaction contribution we have derived the Euler-Lagrange equation for the particle which is just the relativistic generalization of Newton's equation. The force on the particle is the Lorentz force which is relativistically invariant due to the covariance of Maxwell's equations.

What is left now is to consider the influence of the field-matter interaction on the electromagnetic fields. We expect that this will give rise to modified Maxwell's equations compared to the vacuum case - the equations will contain a coupling to the particles via charge and current densities. Since this coupling is, in general, not created by point particles, we first study how to generalize the field-matter interaction Lagrangian to arbitrary spatially extended charge distributions $\rho(\mathbf{r})$.

Generalization to a delocalized charge distribution

To study the interaction of the field with a continuous charge distribution we will first generalize our result to an ensemble of N point particles with total charge $Q = \sum_{i=1}^N e_i$ and then let the individual charges go to zero and increasing N such that Q remains finite. Then we can change from summation to integration over the volume,

$$Q = \sum_{i=1}^{N} e_i \longrightarrow \int_V d^3r\, \rho(\mathbf{r}),$$

and write the total interaction contribution to the action, generalizing Eq. (1.180) according to

$$\begin{aligned}
S_{MF} &= \sum_{i=1}^{N} S_{Mi,F} = -\frac{1}{c}\int_a^b A_\mu \sum_{i=1}^{N} e_i dx^\mu \\
&\longrightarrow -\frac{1}{c}\int_a^b \int_V d^3r\, \rho(\mathbf{r})\, A_\mu\, dx^\mu \\
&= -\frac{1}{c}\int_{t_a}^{t_b} \int_V d^3r\, \rho(\mathbf{r})\, A_\mu \frac{dx^\mu}{dt} dt \equiv \int_{t_a}^{t_b} dt \int_V d^3r\, \mathcal{L}_{MF}(\mathbf{r}, t). \quad (1.190)
\end{aligned}$$

The last line allows us to identify the Lagrange density which can be expressed by the current 4 vector using the definition $j^\mu = \rho\frac{dx^\mu}{dt} = (c\rho, \mathbf{j})$, cf. Eq. (1.111),

$$\boxed{\mathcal{L}_{MF} = -\frac{1}{c}A_\mu j^\mu} \tag{1.191}$$

which is an impressively simple result considering the scope of physical processes it describes. This is the simplest *linear* coupling of the electromagnetic field to charged matter where the latter is fully described by the current vector j^μ. This result is fully applicable to classical systems: in this case we have to insert the charge current density and charge density which contain delta functions which brings us back to the first expression for the action involving a sum over point charge contributions. At the same time, the result (1.191) also applies to quantum systems. Then the charge and current density are determined by the wave function,

$$\begin{aligned}
\rho &= e\,\psi\psi^*, \\
\mathbf{j} &= \frac{e\hbar}{2mi}\left\{\psi^*\nabla\psi - \psi\nabla\psi^*\right\},
\end{aligned}$$

if the system is in the (pure) quantum state described by the wave function $\psi(\mathbf{r}, t)$. Thus, in the quantum case, the current entering the Lagrange density is to be understood as an expectation value. Therefore, any technical tool to compute averages can be applied here. If the quantum system is in a mixed state, e.g. in thermodynamic equilibrium, the expectation value can be computed e.g. via the trace with the density operator, a distribution function or Green's function. Finally, the coupling (1.191) has proved very successful in the description of other field-matter interactions far beyond electrodynamics, including quantum chromodynamics, see e.g. Ref. [GSS07].

1.6.4 Quantization of the electromagnetic field coupled to charges

After using the action of the field-matter system to derive the equations of motion of the particles we now have to do the same for the electromagnetic field. Since we will perform the variations with respect to the field variable A^μ, from the three contributions to the action (1.171) only two will have an effect; the particle term S_M does not depend on the field and can be skipped. Thus the Lagrange density $\mathcal{L}_F$ of the field in vacuum, cf. Sec. 1.5.2 has to be generalized to $\mathcal{L}_F \longrightarrow \mathcal{L}_F + \mathcal{L}_{MF} \equiv \mathcal{L}_{\text{field}}$ which reads

$$\boxed{\mathcal{L}_{\text{field}}[A^\mu, \partial^\nu A^\mu, j^\mu] = -\frac{1}{8\pi}\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)\partial_\mu A_\nu - \frac{1}{c}A_\mu j^\mu} \tag{1.192}$$

In contrast to the vacuum case the Lagrangian now also depends on A^μ. The equations of motion are given by the Euler-Lagrange equations (1.128) in which only the first term is new,

$$\begin{aligned} 0 &= \frac{\delta \mathcal{L}_{\text{field}}}{\delta A_\nu} - \partial_\mu \frac{\delta \mathcal{L}_{\text{field}}}{\delta \partial_\mu A_\nu}, \\ &= -\frac{1}{c} j^\mu + \partial_\mu \frac{F^{\mu\nu}}{4\pi}, \end{aligned} \tag{1.193}$$

and we immediately obtain Maxwell's equations for the field tensor including the coupling to matter. The second pair of equations for the dual tensor does not change compared to the vacuum case since it does not involve charge and current densities.

Let us now discuss how the presence of charges alters the quantization of the electromagnetic field. The first thing we need is to compute the canonical momentum of the field by functionally differentiating $\mathcal{L}_{field}$ with respect to $\partial_t A_\nu$, cf. Eq. (1.131). Since $\mathcal{L}_{MF}$ is independent of $\partial_t A_\nu$ the result is the same as in vacuum, $\pi^0 \equiv 0$ and $\vec{\pi} = \mathbf{E}/4\pi$. This means, the quantization can be performed as in vacuum, by replacing the fields A^μ and $\vec{\pi}$ by operators with the commutation relations

$$\left[\hat{A}^k(\mathbf{r}, t), \hat{\pi}^j(\mathbf{r}', t)\right] = i\hbar\, \delta_{j,k} \delta(\mathbf{r} - \mathbf{r}'). \tag{1.194}$$

Note that we did not use the transverse delta function here because we cannot guarantee that there exist only two transverse modes as in vacuum electrodynamics since the fields are subject to external currents,

$$\Box A^k = \frac{4\pi}{c} j^k, \qquad k = 1, 2, 3. \tag{1.195}$$

Here we have excluded the $0-$component, i.e. the equation for ϕ because it is not independent of the dynamics of $\mathbf{A}$. For example, in the Lorentz gauge, the connection is given by Eq. (1.108). The general solution of this equation is

$$A^k = C_k A^k_{\text{free}} + A^k_{\text{ext}},$$

where A_{free} is the solution of the homogeneous equation (consisting of two purely transverse components) and A_{ext} is the particular solution of the inhomogeneous equation

$$\mathbf{A}_{\text{ext}}(\mathbf{r}, t) = \frac{4\pi}{c} \int d\bar{t} \int d^3\bar{r}\, G(\mathbf{r}, t; \bar{\mathbf{r}}, \bar{t})\, \mathbf{j}(\bar{\mathbf{r}}, \bar{t}),$$

where G is the Green function of Eq. (1.195).

The explicit form of G and of all solutions of Eq. (1.195) depends on the geometry of the system, the boundary values of $\mathbf{j}$ and the associated initial and boundary conditions for $\mathbf{A}$. These solutions can again be used to construct a complete set of normal modes $\mathbf{u}_{\lambda,\mathbf{k}}(\mathbf{r}, t; [\mathbf{j}])$ which, due to linearity of the wave equation, can be used as a basis for the canonically conjugate fields $\mathbf{A}$ and π. The same basis can be used for the field operators $\hat{\mathbf{A}}$ and $\hat{\pi}$, following the procedure outlined in Sec. 1.5.4. Each of the modes will again be associated with a pair of photon operators $\hat{b}_{\lambda,\mathbf{k}}(\mathbf{r}, t)$ and $\hat{b}^{\dagger}_{\lambda,\mathbf{k}}(\mathbf{r}, t)$. With this, the second quantization procedure has been extended to the case of fields coupled to external charges.

1.6.5 Quantization of the EM field in a dielectric medium or plasma

So far we have considered the interaction of the field with external charges and currents and derived the coupled equations of charges and field in various forms, including a quantized description. This is a correct microscopic description However, if the field interacts with a macroscopic number of particles, for example with a plasma, a gas or a polarizable medium, it is advantageous to use a statistical (continuum) approach. The idea is not to resolve the individual charged particle dynamics but to average all quantitities over a finite volume.

This medium itself contains charges and currents which are "induced" by the field giving rise to a total charge and current density $\rho = \rho_{\rm ext} + \rho_{\rm ind}$ and $\mathbf{j} = \mathbf{j}_{\rm ext} + \mathbf{j}_{\rm ind}$ which appear in Maxwell's equations. This leads to a modified "medium" electrodynamics, see e.g. the text books [ABA84, Jac75, LL80, **?**].

Here we only outline the basic procedure. The common approach is to restore equations of the same structure as those of vacuum electrodynamics, cf. Sec. 1.5.1,

$$\mathrm{div}\mathbf{D}(\mathbf{r}, t) = 4\pi\rho_{\rm ext}(\mathbf{r}, t), \quad \nabla \times \mathbf{E}(\mathbf{r}, t) = -\frac{1}{c}\frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t}, \tag{1.196}$$

$$\mathrm{div}\mathbf{B}(\mathbf{r}, t) = 0, \quad \nabla \times \mathbf{H}(\mathbf{r}, t) = \frac{4\pi}{c}\mathbf{j}_{\rm ext}(\mathbf{r}, t) + \frac{1}{c}\frac{\partial \mathbf{D}(\mathbf{r}, t)}{\partial t}. \tag{1.197}$$

keeping only the external charge and currents and absorbing the induced contributions into modified fields $\mathbf{D}$ and $\mathbf{H}$. The relation between $\mathbf{D}$ and $\mathbf{E}$ on the one hand, and $\mathbf{H}$ and $\mathbf{B}$, on the other is realized via additional functions – the dielectric function ϵ and the magnetic permeability μ, respectively which contain the complete information about the particular medium. In the simplest case, the relation in Fourier space reads $(i, j = 1, 2, 3)$,

$$\begin{aligned} D_i(\mathbf{k}, \omega) &= \epsilon_{ij}(\mathbf{k}, \omega) E_j(\mathbf{k}, \omega), \\ B_i(\mathbf{k}, \omega) &= \mu_{ij}(\mathbf{k}, \omega) H_j(\mathbf{k}, \omega), \end{aligned}$$

where summation over repeated indices is implied.

Now, the eigenvalue problem of the field in the presence of a polarizable medium differs compared to vacuum. As a result, for the case of an unmagnetized medium ($\mu = 1$), the eigenmodes of the electromagnetic field are obtained form the condition that the following determinant vanishes, i.e.

$$\left| k^2 \delta_{ij} - k_i k_j - \frac{\omega^2}{c^2} \epsilon_{ij}(\mathbf{k}, \omega) \right| = 0.$$

The result is a spectrum of modes with the dispersion $\omega_s(\mathbf{k}), \quad s = 1, 2, \ldots,$ which are essentially influenced by the medium, for details see [ABA84, Bon98, **?**]. As in the case of the electromagnetic field in vacuum, the details of the modes depend on the geometry of the system. In case that the field is enclosed in a cube, cf. Sec. 1.5.3, the solutions will again be monochromatic plane waves but with a modified frequency dispersion $\omega_s(\mathbf{k})$. After diagonalization, these modes can again be used to construct a complete orthonormal set suitable for expansion of any field excitation. In particular, the canonical fields and the field operators can be expanded in terms of these normal modes.

This is, however, only part of the story. The procedure outlined before and discussed so far leads to the expansion of the field operators $\hat{A}$ and $\hat{\pi}$ in terms of normal modes where the coefficients are creation and annihilation operators $\hat{b}^\dagger$ and $\hat{b}$ of elementary excitations of a given normal mode. However, these are randomly fluctuating operators and, therefore, also $\hat{A}$ and $\hat{\pi}$ are random quantities. The only quantities of practical use which can be related to an experiment are suitable expectation values such as $\langle \hat{A}^\mu \rangle$. Since in many cases this mean value is zero – an exception or Bose systems below the condensation temperature, cf. Sec. 5.4 – the first non-trivial expectation value is that of a product $\langle \hat{A}^\mu \hat{A}^\nu \rangle$ which is closely related to the Green's functions of many-body theory. Thus the task of a many-body theory of field and matter is to derive equations of motion for the Green's functions of the electromagnetic field (photon Green's function) coupled to the Green's functions of charge particles. This is described in detail in Ref. [Bon16]. An alternative way in dealing with the expectation value is to perform a semiclassical averaging over random initial configurations. This is discussed in a number of subsequent sections, in particular in Secs. 2.1.4, 5.9.2.

Finally, we note that the concept of canonical quantization is also applied to the wave equations of quantum mechanics such as the Schrdinger equation or the Klein-Gordon and Dirac equations. This leads to a replacement of the wave functions or spinors by the respective field operators. For more details on this approach, see the text book of Huang [Hua98].

1.7 Problems to Chapter 1

1. Analyze the second variation of the action. Are there cases when the Euler-Lagrange equations (1.18) lead to a maximum of the action S rather than to a minimum?

2. Proof relation (1.70).

3. Derive the hamiltonian (1.97) from the expression (1.96).

4. Verify the components of the tensor $\tilde{F}^{\mu\nu}$ by direct evaluation of the definition, Eq. (1.122).

5. Verify Eq. (1.123) by direct evaluation of the tensor product.

6. Verify Eq. (1.124) by direct evaluation of the tensor product.

7. Derive the second quantization representation of the hamiltonian, Eq. (1.170).

8. Derive the second quantization representation of the Poynting vector, Eq. (1.138).